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Solution exam Linear Algebra on Friday April 18, 2019, 13.45 – 15.45 hours.
Exercises 1,2,5,7 and 9 only receive points based on the final answer.
Exercises 3,4,6 and 8 require a motivated answer.

1.

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Determine A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -2 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 2 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 2 & -3 \end{array}\right)$$

and hence:

$$A^{-1} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & -3 \end{pmatrix}.$$

2.

The matrix A given by:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & -1 & -2 \end{pmatrix}$$

has eigenvalues in -1, 0 and 1. Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

We need to find the eigenvectors of the matrix A. First we find an eigenvector $\mathbf{x}_1$ associated to eigenvalue 1, i.e.

$$A\mathbf{x}_1 = \mathbf{x}_1$$

or equivalently:

$$0 = (A - I)\mathbf{x}_1 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 3 \\ 1 & -1 & -3 \end{pmatrix} \mathbf{x}_1$$

We have:

$$\begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 3 \\ 1 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 3 \\ 1 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

We choose

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Next, we find an eigenvector $\mathbf{x}_2$ associated to eigenvalue 0, i.e.

$$A\mathbf{x}_2 = \mathbf{0}$$

or, equivalently:

$$0 = A\mathbf{x}_2 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & -1 & -2 \end{pmatrix} \mathbf{x}_2$$

We have:

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ -1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We choose

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Finally, we find an eigenvector $\mathbf{x}_3$ associated to eigenvalue -1, i.e.

$$A\mathbf{x}_3 = -\mathbf{x}_3$$

or, equivalently:

$$0 = (A + I)\mathbf{x}_3 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 3 & 3 \\ 1 & -1 & -1 \end{pmatrix} \mathbf{x}_3$$

We have:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 3 & 3 \\ 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 4 \\ 0 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We choose

$$\mathbf{x}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Therefore, we can choose:

$$P = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

3.

> Consider the following three vectors:
>
> $$\mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}.$$
>
> Determine a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

We have:

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 0 \\ -1 & 2 & 4 \\ 2 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since we have pivots in the first two columns, we find:

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

and the required basis is equal to:

$$\{\mathbf{v}_1, \mathbf{v}_2\}$$

4.

> Consider the matrices A and B given by:
>
> $$A = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 5 & -2 \\ -1 & -2 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -2 & -1 \\ 2 & -3 & -2 \\ -1 & 2 & 2 \end{pmatrix},$$
>
> Find all vectors $\mathbf{x} \in \mathbb{R}^3$ for which:
>
> $$A\mathbf{x} = \mathbf{x} \quad \text{and} \quad B\mathbf{x} = \mathbf{x}.$$

$\mathbf{x}$ needs to satisfy:

$$(A - I)\mathbf{x} = \mathbf{0} \quad \text{and} \quad (B - I)\mathbf{x} = \mathbf{0}$$

or equivalently:

$$\mathbf{x} \in \operatorname{Null} \begin{pmatrix} A - I \\ B - I \end{pmatrix} \mathbf{x} = \operatorname{Null} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \\ 1 & -2 & -1 \\ 2 & -4 & -2 \\ -1 & 2 & 1 \end{pmatrix} \mathbf{x}$$

We have:

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \\ 1 & -2 & -1 \\ 2 & -4 & -2 \\ -1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & -8 & 0 \\ 0 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and hence we find that

$$\mathbf{x} \in \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

5.

> Given is that two matrices A and B have the same reduced echelon form. Indicate for each of the following four statements whether they are consequences of the fact that the reduced echelon form of A and B are the same:
>
> 1) $\operatorname{Null} A = \operatorname{Null} B$,
>
> 2) $\operatorname{Col} A = \operatorname{Col} B$,
>
> 3) A invertible implies that B is invertible,
>
> 4) $\det A = \det B$.

The null space is not affected by elementary row operations and hence property 1 is true.

The column space however is clearly affected by row operations. For instance:

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

have the same reduced echelon form but different column space. Hence property 2 is not true.

A is invertible if the associated reduced echelon form has a pivot in every row and in every column. But then B also has a reduced echelon form with a pivot in every row and in every column which implies B is invertible. Hence property 3 is true.

The determinant is clearly affected by row operations. For instance:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

have the same reduced echelon form but different determinants. Hence property 4 is not true.

6.

> Given are two matrices $A, B \in \mathbb{R}^{3\times 3}$ where the columns of A are given by $\mathbf{a}_1, \mathbf{a}_2$ en $\mathbf{a}_3$ while the columns of B are given by $\mathbf{b}_1, \mathbf{b}_2$ en $\mathbf{b}_3$. We have
>
> $$\mathbf{b}_1 = 2\mathbf{a}_1 - \mathbf{a}_3, \qquad \mathbf{b}_2 = 2\mathbf{a}_3 \quad \text{and} \quad \mathbf{b}_3 = 3\mathbf{a}_2 + 2\mathbf{a}_3$$
>
> and $\det A = 3$. Determine $\det B$.

We have:

$$\begin{aligned}
\det B = \det\begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{pmatrix} &= \det\begin{pmatrix}2\mathbf{a}_1 - \mathbf{a}_3 & 2\mathbf{a}_3 & 3\mathbf{a}_2 + 2\mathbf{a}_3\end{pmatrix} \\
&= 2\det\begin{pmatrix}2\mathbf{a}_1 - \mathbf{a}_3 & \mathbf{a}_3 & 3\mathbf{a}_2 + 2\mathbf{a}_3\end{pmatrix} \\
&= 2\det\begin{pmatrix}2\mathbf{a}_1 & \mathbf{a}_3 & 3\mathbf{a}_2\end{pmatrix} \\
&= 12\det\begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_3 & \mathbf{a}_2\end{pmatrix} \\
&= -12\det\begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{pmatrix} = 12\det A = -36
\end{aligned}$$

7.

> $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that rotates each point $(x_1, x_2) \in \mathbb{R}^2$ counter clockwise over 120 degrees. Determine the representation matrix of T.

We have:

$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\sqrt{3}\end{pmatrix}$$

and

$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\end{pmatrix}$$

and therefore:

$$[T] = \begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}$$

8.

> Assume that the solution set of the system $A\mathbf{x} = \mathbf{b}$ has at least two solutions, i.e. we have $\mathbf{x}_1$ and $\mathbf{x}_2$ with $\mathbf{x}_1 \neq \mathbf{x}_2$ such that $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$.
> Show that the system $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions.

If we choose

$$\mathbf{x} = \mathbf{x}_1 + \lambda(\mathbf{x}_1 - \mathbf{x}_2)$$

for an arbitrary constant λ we get:

$$\begin{aligned}
A\mathbf{x} &= A\left[\mathbf{x}_1 + \lambda(\mathbf{x}_1 - \mathbf{x}_2)\right] \\
&= A\mathbf{x}_1 + \lambda(A\mathbf{x}_1 - A\mathbf{x}_2) \\
&= \mathbf{b} + \lambda(\mathbf{b} - \mathbf{b}) \\
&= \mathbf{b}.
\end{aligned}$$

Therefore $\mathbf{x}$ is a solution for any choice of λ which clearly yields an infinite number of solutions.

9.

Given is a matrix A parameterized with a constant α: $$A = \begin{pmatrix} 0 & 0 & \alpha-1 \\ 2\alpha-5 & 6-2\alpha & 0 \\ \alpha-3 & 4-\alpha & 0 \end{pmatrix}$$ Determine all $\alpha \in \mathbb{R}$ for which A is invertible.

We have:

$$\begin{aligned} \det A &= (\alpha-1)\det\begin{pmatrix} 2\alpha-5 & 6-2\alpha \\ \alpha-3 & 4-\alpha \end{pmatrix} \\ &= (\alpha-1)\left[(2\alpha-5)(4-\alpha)-(6-2\alpha)(\alpha-3)\right] = (\alpha-1)(\alpha-2) \end{aligned}$$

which yields that A is invertible for all $\alpha \in \mathbb{R}$ with $\alpha \neq 1$ and $\alpha \neq 2$.