UNIVERSITY OF TWENTE

Department of Electrical Engineering, Mathematics and Computer Science

Solution exam Linear Algebra on Friday March 29, 2019, 13.45 – 15.45 hours.

Exercises 1,2,4,8 and 9 only receive points based on the final answer.
Exercises 3,5,6 and 7 require a motivated answer.

1.

Consider the function f given by $f(x) = \alpha e^x + \beta e^{2x} + \gamma e^{-x}$, where α, β and γ are unknown constants.
Determine α, β and γ if it is given that

$$f(0) = -1, \quad f'(0) = 5, \quad f''(0) = 5.$$

We obtain:

$$\begin{aligned} f(0) &= \alpha + \beta + \gamma = -1 \\ f'(0) &= \alpha + 2\beta - \gamma = 5 \\ f''(0) &= \alpha + 4\beta + \gamma = 5 \end{aligned}$$

The augmented matrix associated to the linear system is:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & -1 & 5 \\ 1 & 4 & 1 & 5 \end{pmatrix}$$

Using elementary row operations we get:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & -1 & 5 \\ 1 & 4 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 6 \\ 0 & 3 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 6 & -12 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

The simplified system is:

$$\begin{aligned} \alpha &= -1 \\ \beta &= 2 \\ \gamma &= -2 \end{aligned}$$

which yields the requested answer.

2.

The matrix A is given by:

$$A = \begin{pmatrix} -3 & 0 & -1 \\ 1 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

a) Determine $\det A$.

Expanding along the second column we get:

$$\det A = 1 \cdot \det \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix} = -3 + 5 = 2.$$

b) A has eigenvalue 1 (you don't have to prove this).
Determine the corresponding eigenspace.

We need to find all $\mathbf{x} \neq 0$ such that

$$A\mathbf{x} = \mathbf{x}$$

or

$$(A - I)\mathbf{x} = \mathbf{0}$$

We have:

$$A - I = \begin{pmatrix} -4 & 0 & -1 \\ 1 & 0 & 2 \\ 5 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 0 & 0 \\ -4 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ -4 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which yields:

$$E_1 = \mathrm{Null}(A - I) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

b) Determine the other (possibly complex) eigenvalue(s) of A.

We have the following characteristic equation:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} -3-\lambda & 0 & -1 \\ 1 & 1-\lambda & 2 \\ 5 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda) \det \begin{pmatrix} -3-\lambda & -1 \\ 5 & 1-\lambda \end{pmatrix}$$
$$= (1-\lambda)\left[(-3-\lambda)(1-\lambda) + 5\right] = (1-\lambda)(\lambda^2 + 2\lambda + 2)$$

Therefore the eigenvalues are given by

$$\lambda = 1 \quad \text{or} \quad \lambda^2 + 2\lambda + 2 = 0$$

which yields that the other two eigenvalues are $-1 + i$ and $-1 - i$.

3.

Consider the matrices

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and } C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Verify whether $A^3B^5C^7$ is invertible.

We have:

$$\det A = 1 + 1 = 2, \quad \det B = 3 - 4 = -1, \quad \det C = 4 - 4 = 0$$

But then:

$$\det(A^3B^5C^7) = (\det A)^3(\det B)^5(\det C)^7 = 2^3(-1)^5 0^7 = 0$$

Therefore $A^3B^5C^7$ is not invertible.

4.

Given is the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ -1 & 2 & -5 \end{pmatrix}.$$

a) Determine Null A.

We have:

$$A \sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 0 & 4 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

which yields:

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

It is then easily seen that:

$$\text{Null}\, A = \text{Span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

b) Determine Col A.

We have the following reduced echelon form for A:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

which has pivots in the first two columns. Hence Col A is the span of the first two columns of the matrix A:

$$\text{Col}\, A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$$

5.

> Consider a vector space and two vectors in $\mathbb{R}^3$:
>
> $$\mathcal{T} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix}.$$
>
> Determine whether $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$, is a basis for $\mathcal{T}$.

We basically need to verify two things. Firstly, whether the vectors in $\mathcal{B}$ are linearly independent. But this is easily seen since $\mathbf{v}_1$ and $\mathbf{v}_2$ are clearly not linear multiples of each other. Secondly we need to verify whether

$$\operatorname{Span}\mathcal{B} = \mathcal{T}$$

$\mathcal{T}$ is spanned by two vectors which we call $\mathbf{t}_1$ and $\mathbf{t}_1$. Note that

$$\operatorname{Span}\mathcal{B} \subset \mathcal{T}$$

if the matrix with columns $\mathbf{t}_1, \mathbf{t}_2, \mathbf{v}_1, \mathbf{v}_2$ has only pivots in the first columns since then both $\mathbf{v}_1$ and $\mathbf{v}_2$ can be expressed in terms of $\mathbf{t}_1$ and $\mathbf{t}_2$. We have:

$$\begin{pmatrix} 1 & 2 & -1 & -3 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & -3 & 3 & 6 \\ 0 & -2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly, we only have pivots in the first two columns and hence $\operatorname{Span}\mathcal{B} \subset \mathcal{T}$. Remains to show that

$$\mathcal{T} \subset \operatorname{Span}\mathcal{B}.$$

Again we see that this is true if the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{t}_1, \mathbf{t}_2$ has only pivots in the first columns since then both $\mathbf{t}_1$ and $\mathbf{t}_2$ can be expressed in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. We have:

$$\begin{pmatrix} -1 & -3 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & -2 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & -3 & 1 & 2 \\ 0 & -3 & 3 & 3 \\ 0 & -2 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & -3 & 1 & 2 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly, we only have pivots in the first two columns and hence $\operatorname{Span}\mathcal{B} \subset \mathcal{T}$. Hence $\mathcal{B}$ is a basis for $\mathcal{T}$.

6.

> Determine two different echelon forms for the following matrix:
>
> $$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

We have:

$$A \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

The last matrix is clearly an echelon form but not a reduced echelon form. We can simply take one more step towards the reduced echelon form and get:

$$A \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and we obtain a second echelon form for the matrix A which is clearly different from the first one.

7.

> Given are four vectors in $\mathbb{R}^3$:
>
> $$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$
>
> In this case $\mathcal{T} = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$ is a basis for $\mathbb{R}^3$. Determine $[\mathbf{v}_4]_{\mathcal{T}}$.

We get:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

and hence:

$$[\mathbf{v}_4]_{\mathcal{T}} = \tfrac{1}{2} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

8.

> Let T be an arbitrary linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.
>
> a) Assume that T is one-to-one. Show that T^2 is one-to-one.

Assume $T^2\mathbf{x} = \mathbf{0}$, i.e. $T(T\mathbf{x}) = \mathbf{0}$. Since T is one-to-one we find $T\mathbf{x} = \mathbf{0}$. But using that T is one-to-one again we find that $\mathbf{x} = \mathbf{0}$. This yields that T^2 is one-to-one.

> b) Assume that T^2 is one-to-one. Show that T is one-to-one.

Assume $T\mathbf{x} = \mathbf{0}$. This clearly implies that $T^2\mathbf{x} = \mathbf{0}$. But then using that T^2 is one-to-one we find that $\mathbf{x} = \mathbf{0}$. This yields that T is one-to-one.

9.

$T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that:

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$$

Determine the representation matrix of T.

Note that

$$\begin{pmatrix}1\\0\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}$$

We compute the image of the standard basis vectors. We get, using linearity:

$$T\begin{pmatrix}1\\0\end{pmatrix} = 2 \cdot T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}1\\2\end{pmatrix} = 2\begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\3\\3\end{pmatrix}$$

and

$$T\begin{pmatrix}0\\1\end{pmatrix} = T\begin{pmatrix}1\\2\end{pmatrix} - T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-2\end{pmatrix}$$

We find:

$$[T] = \begin{pmatrix}-1 & 2\\3 & -1\\3 & -2\end{pmatrix}.$$