UNIVERSITY OF TWENTE
Department of Electrical Engineering, Mathematics and Computer Science

Solution exam Mathematics C1 Cayley on Monday July 6, 2018, 13.45 – 15.45 hours.

Exercises 1,2,3,4,6a,8 and 9 only receive points based on the final answer.
Exercises 5,6b and 7 require a motivated answer.

1.

> Given is the following matrix equation with unknown $X \in \mathbb{R}^{2\times 2}$:
>
> $$A^{\mathrm{T}}X + XA = B,$$
>
> where
>
> $$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$
>
> The unknown matrix X has the following form:
>
> $$X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$$
>
> This leads to a set of linear equations in x_1, x_2 and x_3.
> Determine all possible x_1, x_2 and x_3 for which the above matrix equation is satisfied.

We obtain:

$$AX + XA^{\mathrm{T}} = \begin{pmatrix} 2x_1 + 2x_2 & x_1 + 2x_2 + x_3 \\ x_1 + 2x_2 + x_3 & 2x_2 + 2x_3 \end{pmatrix}$$

This leads to the following set of linear equations:

$$\begin{array}{rcrcrcl} 2x_1 & + & 2x_2 & & & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & = & 2 \\ & & 2x_2 & + & 2x_3 & = & 3 \end{array}$$

and solving this leads to:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \begin{pmatrix} -1 \\ \frac{3}{2} \\ 0 \end{pmatrix} + \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

or

$$X = \begin{pmatrix} -1 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

for any $\alpha \in \mathbb{R}$.

2.

The matrix A is given by:

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & -3 & 2 \\ 3 & -4 & 1 \end{pmatrix}$$

a) Determine a basis for $\operatorname{Null} A$.

We have:

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & -3 & 2 \\ 3 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We obtain:

$$\operatorname{Null} A = \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and a basis for $\operatorname{Null} A$ is given by:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

b) Determine a basis for $\operatorname{Col} A$.

From the reduced echelon form obtained in part a, we immediately see that the first two columns of A are linearly independent. Hence:

$$\operatorname{Col} A = \operatorname{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix} \right\}$$

and a basis for $\operatorname{Col} A$ is given by:

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix} \right\}.$$

3.

Given are four vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

In this case,

$$\mathcal{T} = \operatorname{Span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$$

is a basis for $\mathbb{R}^3$. Determine $[\mathbf{v}_4]_{\mathcal{T}}$.

We get:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

and hence:

$$[\mathbf{v}_4]_{\mathcal{T}} = \tfrac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

4.

Given is the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Determine, if it exists, the inverse of the matrix A.

We have:

$$A^{-1} = -\tfrac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$$

5.

The distance between two vectors in $\mathbb{R}^3$ is given by:

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Given are two vectors:

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} \alpha \\ 2 \\ -1 \end{pmatrix}$$

We look at the set $\mathcal{X}$ of all vectors for which the distance to $\mathbf{v}$ is equal to the distance to $\mathbf{w}$. In other words:

$$\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x} - \mathbf{v}\| = \|\mathbf{x} - \mathbf{w}\| \right\}$$

a) Show that $\mathcal{X}$ is a linear subspace of $\mathbb{R}^3$ when $\alpha = 1$ or $\alpha = -1$.

We need:

$$\|\mathbf{x} - \mathbf{v}\| = \|\mathbf{x} - \mathbf{w}\|$$

or, equivalently:

$$\|\mathbf{x} - \mathbf{v}\|^2 = \|\mathbf{x} - \mathbf{w}\|^2$$

Working this out we get

$$(x_1 - 2)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 = (x_1 - \alpha)^2 + (x_2 - 2)^2 + (x_3 + 1)^2$$

which yields:

$$-4x_1 - 2x_2 - 2x_3 + 6 = -2\alpha x_1 - 4x_2 + 2x_3 + 5 + \alpha^2$$

Since $\alpha^2 = 1$ this reduces to:

$$(-4 + 2\alpha)x_1 + 2x_2 - 4x_3 = 0 \qquad \text{2 points}$$

which is one linear equation. Sufficient to note that this is a null space and hence is a linear subspace (1 point). Or you just verify:

$\mathcal{X} \neq \varnothing$, because $\mathbf{0} \in \mathcal{X}$.

If $\mathbf{u}, \mathbf{v} \in \mathcal{X}$ then $\mathbf{u} + \mathbf{v} \in \mathcal{X}$, because:

$$(-4 + 2\alpha)(u_1 + v_1) + 2(u_2 + v_2) - 4(u_3 + v_3)$$
$$= ((-4 + 2\alpha)u_1 + 2u_2 - 4u_3) + (-4 + 2\alpha)v_1 + 2v_2 - 4v_3) = 0$$

If $\mathbf{u} \in \mathcal{X}$ and $\beta \in \mathbb{R}$ then $\beta\mathbf{u} \in \mathcal{X}$, because:

$$(-4 + 2\alpha)(\beta u_1) + 2(\beta u_2) - 4(\beta u_3) = \beta((-4 + 2\alpha)u_1 + 2u_2 - 4u_3) = 0$$

b) Show that $\mathcal{X}$ is **not** a linear subspace of $\mathbb{R}^3$ when $\alpha \neq 1$ and $\alpha \neq -1$.

For $\mathcal{X}$ to be a linear subspace, we need that $\mathbf{0} \in \mathcal{X}$. This requires:

$$\|\mathbf{0} - \mathbf{v}\| = \|\mathbf{0} - \mathbf{w}\|$$

or, equivalently:

$$\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2$$

Working this out we get

$$2^2 + 1^2 + 1^2 = \alpha^2 + 2^2 + (-1)^2$$

which clearly requires that $\alpha^2 = 1$ and we know this is not true. Therefore $\mathcal{X}$ is not a linear subspace.

6.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which takes each point $(x_1, x_2) \in \mathbb{R}^2$ and rotates it first through 45 degrees (counterclockwise) and then projects the result on the line $y = x$.

a) Determine the representation matrix of T.

We compute the image of the standard basis vectors. We get

$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\frac{1}{2}\sqrt{2}\\ \frac{1}{2}\sqrt{2}\end{pmatrix} \to \begin{pmatrix}\frac{1}{2}\sqrt{2}\\ \frac{1}{2}\sqrt{2}\end{pmatrix} \qquad \text{1 point}$$

and

$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\frac{1}{2}\sqrt{2}\\ \frac{1}{2}\sqrt{2}\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix} \qquad \text{1 point}$$

where in each case I first rotate and then project. I find:

$$[T] = \begin{pmatrix}\frac{1}{2}\sqrt{2} & 0\\ \frac{1}{2}\sqrt{2} & 0\end{pmatrix} \qquad \text{1 point}$$

b) Determine whether T is surjective (onto) and/or injective (one-to-one).

Since T maps any point on the line $y = x$, it can never be surjective since all points outside of this line can never be reached (1 point).

In part a) we have seen that the vector

$$\begin{pmatrix}0\\1\end{pmatrix}$$

is mapped to **0**. Therefore the transformation is clearly not injective (1 point).

7.

You are given that the matrix A is invertible and diagonalizable. Show that A^{-1} is also diagonalizable.

Since A is diagonalizable there exists an invertible matrix P and a diagonal matrix D such that:

$$P^{-1}AP = D \qquad \text{or} \qquad A = PDP^{-1} \qquad \text{1 point}$$

Since A is invertible, we know that D is invertible (1 point). But then:

$$P^{-1}A^{-1}P = D^{-1} \qquad \text{or} \qquad A^{-1} = PD^{-1}P^{-1} \qquad \text{1 point}$$

which shows that A^{-1} is diagonalizable since clearly D^{-1} as the inverse of a diagonal matrix is still diagonal.

8.

Determine the volume of the parallelepiped with vertices $(0,0,0)$, $(0,-2,2)$, $(-5,0,-4)$ en $(3,0,4)$.

This is equal to

$$\left|\det\begin{pmatrix}0 & -5 & 3\\ -2 & 0 & 0\\ 2 & -4 & 4\end{pmatrix}\right| = 2\left|\det\begin{pmatrix}-5 & 3\\ -4 & 4\end{pmatrix}\right| = 2|-20+12| = 16$$

Hence the volume is equal to 16. 3 points.

9.

Consider the matrix

$$A = \begin{pmatrix} 1-2\alpha & -2\alpha & \alpha \\ \alpha+1 & \alpha+2 & -1 \\ 2-2\alpha & 2-2\alpha & 2\alpha-1 \end{pmatrix}$$

with $\alpha \in \mathbb{R}$. Determine all $\alpha \in \mathbb{R}$ for which A has eigenvalue 1 with a corresponding eigenspace with dimension 1.

Since we are interested in eigenvectors for eigenvalue 1 we consider

$$\text{Null}(A-I) = \text{Null}\begin{pmatrix} -2\alpha & -2\alpha & \alpha \\ \alpha+1 & \alpha+1 & -1 \\ 2-2\alpha & 2-2\alpha & 2\alpha-2 \end{pmatrix}$$

and we looking for $\alpha \in \mathbb{R}$ for which this subspace is one-dimensional.

If $\alpha \neq 1$ and $\alpha \neq 0$ then we can divide the first row by α and divide the third row by $2-2\alpha$ and we get:

$$\begin{pmatrix} -2\alpha & -2\alpha & \alpha \\ \alpha+1 & \alpha+1 & -1 \\ 2-2\alpha & 2-2\alpha & 2\alpha-2 \end{pmatrix} \sim \begin{pmatrix} -2 & -2 & 1 \\ \alpha+1 & \alpha+1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & \alpha \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which, given two pivots, clearly shows the null-space is 1 dimensional. For $\alpha = 1$ we get:

$$\begin{pmatrix} -2 & -2 & 1 \\ 2 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which clearly shows the null-space is 2 dimensional. For $\alpha = 0$ we get:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which clearly shows the null-space is 2 dimensional.

Therefore the answer is: $\alpha \neq 1$ and $\alpha \neq 0$.