Solutions Introduction to Mathematics Sample Test 1

1. [2pt]
$A = \{\frac{k}{k+1} | k \in \mathbb{N}\}$.
$\inf A = \frac{1}{2}$
$\max A$ does not exist
$\min A = \frac{1}{2}$
$\sup A = 1$

2. [2pt]
$p \wedge p$ is neither tautology nor contradiction.
$p \wedge \neg p$ is a contradiction.
$p \vee \neg p$ is: a tautology.
$p \rightarrow \neg p$ is neither tautology nor contradiction.

3. [3 pt]
Let $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. We prove the statement: If the product mn is even, then either m is even, or n is even (or both).
Assume for contradiction that the statement is false. Then there exist $m, n \in \mathbb{Z}$ for which mn is even, but both m and n are not even.
Then m and n are both odd: $\exists x(m = 2x + 1)$ and $\exists y(n = 2y + 1)$.
Now, $mn = (2x + 1)(2y + 1) = 4xy + 2x + 2y + 1 = 2(2xy + x + y) + 1$,
which is odd, since $2xy + x + y \in \mathbb{Z}$.
This contradicts the fact that mn is even. Therefore our assumption must be false.
We conclude that the statement is true.

4. [3 pt]
Consider statement $S(n) : \sum_{i=0}^{n} \binom{i+2}{i} = \binom{n+3}{n}$ We need to use induction to prove that the statement holds $\forall n \in \mathbb{N} \cup \{0\}$.

Basis Step (n=0)
$\sum_{i=0}^{0} \binom{i+2}{i} = \binom{2}{0} = \frac{2!}{0!2!} = 1.$
$\binom{0+3}{0} = \frac{3!}{0!3!} = 1,$
Since these values are equal, $S(0)$ holds.

Induction Step
Suppose $S(k)$ holds for some $k \in \mathbb{N} \cup \{0\}$, that is $\sum_{i=0}^{k} \binom{i+2}{i} = \binom{k+3}{k}$ (IH).
We need to show that $S(k+1) : \sum_{i=0}^{k+1} \binom{i+2}{i} = \binom{(k+1)+3}{k+1}$ holds, using the following hint: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$ for all $n \in \mathbb{N}, r \in \mathbb{N}, r \leq n$.
Well,

$$\begin{aligned}\sum_{i=0}^{k+1} \binom{i+2}{i} &= \left(\sum_{i=0}^{k} \binom{i+2}{i}\right) + \binom{(k+1)+2}{k+1} \\ &= \binom{k+3}{k} + \binom{k+3}{k+1} \\ &= \binom{k+4}{k+1},\end{aligned}$$

which is what we needed to prove. Here the second equality follows from the induction hypothesis, and the third equality follows from the hint, since $k+3 \geq k$.
By mathematical induction, $S(n)$ holds for all $n \in \mathbb{N} \cup \{0\}$.

5.
Consider the set A of numbers consisting of 4 digits, where each digit is from the set {1,2,3}.

(a) [1 pt] The final digit of any odd number in A is 1 or 3. We have 2 possibilities for the last digit, and 3 possibilities for each of the other 3 digits.
By the product rule, there are $2 \cdot 3^3 = 54$ odd numbers in A.

(b) [2 pt] As shown in part a), there are $2 \cdot 3^3 = 54$ odd numbers in A. Denote the set of odd numbers in A by B.
If a number starts with the digit 1, then there are $3^3 = 27$ possibilities for the remaining digits. Denote the set of numbers in A starting with the digit 1 by C.
If a number starts with the digit 1, and is odd, then there are 2 possibilities for the last digit, and 3 possibilities for digits 2 and 3, which yields a total of $2 \cdot 3^2 = 18$ possibilities. Therefore $|B \cap C| = 18$.
Now, by the principle of inclusion/exclusion, the amount of numbers in A that are are either odd, or start with the digit 1 (or both) equals:
$|B| + |C| - |B \cap C| = 54 + 27 - 18 = 63.$