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UNIVERSITY OF TWENTE.

Course : Introduction to Mathematics & Calculus 1A
Module : 1A
Course code : 2020011[87–93]

Date : November 17, 2025
Time : 08:45–11:45
Reference : Test 2

C

Introduction to Mathematics & Calculus 1A

Instructions

- For questions 1–5, you are only required to fill in the final answer on the answer form.
- For questions 6–12, you are required to write down a full calculation and argumentation.

Please hand in your answer form only. If you run out of space, you can use the extra space at the end of the answer form. Refer clearly to that space in the original answer.
Do not write with red pen or pencil. Do not use correction fluid or tape.
The use of electronic devices is not allowed!

Final answer questions

Report all your answers on the answer form.

1. Consider the intervals $A_k = [-k, k]$ for $k \in \{1, \dots, 80\}$. [2 pt]
Determine $\bigcup_{k=1}^{80} A_k$ and $\bigcap_{k=1}^{80} A_k$.

2. Consider the statement $S_1 : \neg\,(\forall x > 0 (f(x) = 0 \rightarrow g(x) = 0))$ involving arbitrary functions f and g from $\mathbb{R}$ to $\mathbb{R}$. Which of the following statements are logically equivalent to S_1 for all such functions f, g? [2 pt]
There could be zero, one, or multiple equivalent statements.

- **(a)** $\neg\,(\forall x > 0\,(g(x) = 0 \rightarrow f(x) = 0))$
- **(b)** $\neg\,(\forall x > 0\,(g(x) \neq 0 \rightarrow f(x) \neq 0))$
- **(c)** $\exists x > 0\,(\neg(f(x) = 0 \rightarrow g(x) = 0))$
- **(d)** $\exists x \leq 0\,(f(x) = 0 \rightarrow g(x) = 0)$
- **(e)** $\forall x > 0\,(\neg(f(x) = 0 \rightarrow g(x) = 0))$
- **(f)** $\forall x \leq 0\,(f(x) = 0 \rightarrow g(x) = 0)$

3. On your answer sheet, a vector $\mathbf{v}$ is drawn in the xy-plane. [2 pt]
In the same diagram, shade in the points (x, y) for which $(x\mathbf{i} + y\mathbf{j}) \bullet \mathbf{v} \leq 0$.

4. Let $f\colon \mathbb{R} \rightarrow \mathbb{R}$ be a function. Assume that f has derivatives of all orders and satisfies the following properties: $f(1) = 2$, $f'(1) = 6$, $f''(1) = 10$, $f'''(1) = 9$, and

$$\text{for all } x \in \mathbb{R}: \quad |f''(x)| \leq 13 \quad \text{and} \quad |f'''(x)| \leq 17.$$

Let $P_2(x)$ be the Taylor polynomial of order 2 generated by f at $x = 1$. Let $R_2(x) = f(x) - P_2(x)$ be the remainder term.

- **(a)** Find $P_2(x)$. [1 pt]
- **(b)** The line tangent to the graph of f at $x = 1$ passes through the point $(0, y)$. Find y. [1 pt]
- **(c)** What is the strongest conclusion you can draw, based on the information provided? [2 pt]

(i) $|R_2(1.1)| \leq \frac{9}{6} \times 0.001$ *(ii)* $|R_2(1.1)| \leq \frac{13}{6} \times 0.001$ *(iii)* $|R_2(1.1)| \leq \frac{17}{6} \times 0.001$
(iv) $|R_2(1.1)| \leq \frac{9}{2} \times 0.01$ *(v)* $|R_2(1.1)| \leq \frac{13}{2} \times 0.01$ *(vi)* $|R_2(1.1)| \leq \frac{17}{2} \times 0.01$

Exam continues on the following page.

5. Consider the points $P(-3,2,-1)$ and $Q(3,2,2)$. Let ℓ be the line passing through P and Q.

(a) Find a parametrization of the line ℓ. [1 pt]

(b) Give an equation of the plane which passes through the point $R(2,3,0)$ and is perpendicular to the line ℓ. [1 pt]

Open questions

The full solutions to questions 6–12 must be clearly written down on the answer form, including calculations and argumentations.

6. Either prove the following statement or give a counterexample: [3 pt]

For all $a,b,c \in \mathbb{Z}$: If ($a+b$ is even and $b+c$ is even), then $a+c$ is even.

7. Use mathematical induction on n to prove the following statement: [4 pt]

For all $n \in \mathbb{N}$: $8^n - 3^n$ is divisible by 5.

8. Let A be the set of numbers with 7 digits, for which all digits are from $\{1,2\}$. For example, $2212211 \in A$.

(a) How many numbers in A are odd? [1 pt]

(b) How many numbers in A are odd and/or consist of exactly three digits 1 (and therefore four digits 2) in any order? [2 pt]

9. The function f is defined as $f(x) = \begin{cases} \frac{\sin^2(x)-2x}{x} & \text{if } x > 0 \\ ax+b & \text{if } x \leq 0. \end{cases}$

Here, a and b are unspecified real numbers.

(a) Find all values of a and b for which f is continuous at $x=0$. [1 pt]

(b) *Using the limit definition of the derivative*, find all values of a and b for which f is differentiable at $x=0$. [2 pt]

10. Compute the following limits:

(a) $\lim\limits_{x\to 2} \dfrac{x-5}{x^2-2}$ [1 pt]

(b) $\lim\limits_{x\to\infty} \dfrac{5x^3-2x+7}{-7x^3+2x^2+2}$ [1 pt]

(c) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x}-x\right)$ [2 pt]

11. For $a > 4$, the line through $A(a,0)$ and $P(4,\frac{1}{2})$ intersects the y-axis in $B(0,b)$.
Let T be the area of the triangle formed by A, and B, and the origin O.

(a) Show that $T = \frac{a^2}{4(a-4)}$. [1 pt]

(b) Find the value of $a \in [5,10]$ that minimises T. [3 pt]

12. Let the function $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \ln\left((x-1)^2+4\right)$.

(a) Find the derivative of f. [1 pt]

(b) Find the point(s) of inflection of f, and the intervals on which f is concave up/concave down. [2 pt]

Total: 36 pt