Solutions for Discrete Mathematics for Computer Science Exam, Part 2
October 27, 2021

Provide explanation for all answers. Failing to do this would result in no points given
Use of electronic devices is not allowed
Answers to different problems must go in separate sheets
Time: one hour

1. (3 pts.) Prove by mathematical induction that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

Sol. We proceed by mathematical induction. Call the property above $P(n)$

(a) Base case, $n = 1$ **(1.0pts.)**:

$$L.H.S. : \sum_{k=1}^{1} \frac{1}{k^2} = 1$$
$$R.H.S. : 2 - 1$$

$\therefore$ L.H.S $\leq$ R.H.S., so $P(1)$ is true.

Inductive case **(2.0pts.)**. Assume the induction hypothesis for particular m, meaning that $P(m)$ is true. Then we consider the $(m+1)$-case.
On the L.H.S we have

$$\begin{aligned}
P(m+1) : \sum_{k=1}^{m+1} \frac{1}{k^2} &= \sum_{k=1}^{m} \frac{1}{k^2} + \frac{1}{(m+1)^2} \\
&\overset{P(m)}{\leq} 2 - \frac{1}{m} + \frac{1}{(m+1)^2} \\
&\leq 2 - \frac{m+1}{m(m+1)} + \frac{1}{m(m+1)} \\
&= 2 - \frac{m}{m(m+1)} \\
&= 2 - \frac{1}{m+1}.
\end{aligned}$$

Hence, the statement $P(m+1)$ also holds true, establishing our induction step.
Finally by mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.
Note: The solution must clearly indicate where the induction hypothesis was used. This counts for 0.5pts. of the solution.

2. (3 pts.) Consider the function

$$\begin{aligned}
f : \mathcal{P}(\mathcal{U}) \times \mathcal{P}(\mathcal{U}) &\mapsto \mathcal{P}(\mathcal{U}) \\
(A, B) &\mapsto A \cap B.
\end{aligned}$$

(a) Is f one-to-one? Prove it, or provide a counterexample.

(b) Is f onto? Prove it, or provide a counterexample.

Note: If you provide a counterexample, you must specify a universe $\mathcal{U}$ for it.

Sol.

(a) **(1.5pts.)** Consider the sets $A_1, B_1, A_2, B_2 \in P(\mathcal{U})$ such that $A_1 = \{x, y\}, B_1 = \{p, q\}$ and $A_2 = \{r, s\}, B_2 = \{u, v\}$, with x, y, p, q, r, s all different. Then,

$$\begin{aligned} & f(A_1, B_1) = A_1 \cap B_1 = \emptyset, \qquad f(A_2, B_2) = A_2 \cap B_2 = \emptyset \\ \implies \quad & f(A_1, B_1) = f(A_2, B_2) \text{ and } (A_1, B_1) \neq (A_2, B_2). \end{aligned}$$

Therefore, the function f is not injective.

(b) **(1.5pts.)** Let $A \subseteq \mathcal{P}(\mathcal{U})$, then $f(A, A) = A$, and therefore all the elements in $\mathcal{P}(\mathcal{U})$ can be mapped to a set in the domain. Hence, we proved that f is onto.

3. (4 pts.) The following is a false theorem with a wrong proof.

Theorem. Let A be a set and $\mathcal{R}$ a relation on A. If $\mathcal{R}$ is symmetric and transitive, then $\mathcal{R}$ is reflexive.

Proof. Let $(x, y) \in \mathcal{R}$. By symmetry, $(y, x) \in \mathcal{R}$. Next, since $(x, y), (y, x) \in \mathcal{R}$, by transitivity we have $(x, x) \in \mathcal{R}$. Consequently, $\mathcal{R}$ is reflexive.

(a) Indicate a step of the proof that is wrong and why.

(b) Provide a concrete counterexample of a set A and relation $\mathcal{R}$ so that the theorem is false.

Sol.

(a) **(2.0pts.)** First, the assumption of the existence of $(x, y) \in \mathcal{R}$ is wrong. Second, the "proof" only establishes that $(x, x) \in \mathcal{R}$ for a single $x \in A$, whereas reflexivity requires this to hold for all $x \in A$.

(b) **(2.0pts.)** Let $A = \{1\}$ and $\mathcal{R} = \emptyset$. Symmetry and transitivity for $\mathcal{R}$ are satisfied since the relation is empty, but since also $(1, 1) \notin \mathcal{R}$, then this relation is not reflexive.

The End.