Discrete Mathematics for Computer Science Exam, Part 2 October 27, 2021

Provide explanation for all answers. Failing to do this would result in no points given
Use of electronic devices is not allowed
Answers to different problems must go in separate sheets
Time: one hour

1. (3 pts.) Prove by mathematical induction that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

2. (3 pts.) Consider the function

$$\begin{aligned} f : \mathcal{P}(\mathcal{U}) \times \mathcal{P}(\mathcal{U}) &\mapsto \mathcal{P}(\mathcal{U}) \\ (A, B) &\mapsto A \cap B. \end{aligned}$$

   (a) Is f one-to-one? Prove it, or provide a counterexample.

   (b) Is f onto? Prove it, or provide a counterexample.

   Note: If you provide a counterexample, you must specify a universe $\mathcal{U}$ for it.

3. (4 pts.) The following is a false theorem with a wrong proof.

   Theorem. Let A be a set and $\mathcal{R}$ a relation on A. If $\mathcal{R}$ is symmetric and transitive, then $\mathcal{R}$ is reflexive.

   Proof. Let $(x, y) \in \mathcal{R}$. By symmetry, $(y, x) \in \mathcal{R}$. Next, since $(x, y), (y, x) \in \mathcal{R}$, by transitivity we have $(x, x) \in \mathcal{R}$. Consequently, $\mathcal{R}$ is reflexive.

   (a) Indicate a step of the proof that is wrong and why.

   (b) Provide a concrete counterexample of a set A and relation $\mathcal{R}$ so that the theorem is false.