Solutions for Discrete Mathematics for Computer Science Exam, Part 1
October 27, 2021

Provide explanation for all answers. Failing to do this would result in no points given. Use of electronic devices is not allowed. **Answers to different problems must go in separate sheets** Time: one hour

1. (4 pts.) Write the statements below in formal logic:

   (a) Recall that $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers. Consider an open statement $p(n)$, where the free variable n is a nonnegative integer. The statement is:

   If $p(0)$ is true, and for any nonnegative integer k, $p(k)$ is true implies $p(k+1)$ is also true, then $p(n)$ is true for all nonnegative integers n.

   Sol. The solution for part(a) is as follows:

   $p(x) : p(x)$ is true

   (**0.5pts.**) $p(0)$
   (**1.0pts.**) $\forall k \in \mathbb{N}, p(k) \to p(k+1)$

   (**0.5pts.**) $\therefore \forall n \in \mathbb{Z}_+, p(n)$

   Alternatively, the above solution can be condensed as:

   $$[p(0) \wedge (\forall k \in \mathbb{N},\, p(k) \to p(k+1))] \implies (\forall n \in \mathbb{Z}_+\, p(n))$$

   (b) Let L be a set of subsets from a universe $\mathcal{U}$, i.e., $L \subseteq \mathcal{P}(\mathcal{U})$. The statement is

   Each pair of elements from L is either disjoint or one is contained in the other.

   Sol. The solution for part (b) is as follows:

   (**0.5pts.**) $p(X) : (\forall A, B \in X)\, A \cap B = \emptyset$
   (**0.5pts.**) $q(X) : (\forall A, B \in X) : (A \subseteq B) \vee (B \subseteq A)$

   (**1.0pts.**) $(\forall X \subseteq L)\; p(X) \vee q(X)$

   Alternatively, we may also say that,

   $$(\forall A, B \subseteq L) \quad (A \cap B = \emptyset) \vee (A \subseteq B) \vee (B \subseteq A).$$

2. (3 pts.) Using the rules of logic and inference, establish the validity or invalidity of the following argument:

   (1) $p \to q$
   (2) $\neg p \to r$
   (3) $r \to s$
   (4) $\neg q$

   $\therefore s$

Sol. The explanation is as follows:

$$
\begin{array}{rll}
(5) & (1{+}\mathrm{L}13): & \neg q \to \neg p \\
(6) & \text{Premise } (2): & \neg p \to r \\
(7) & (4+5+R1): & \neg p \\
(8) & (7+6+R1): & r \\
(9) & \text{Premise } (3): & r \to s \\
(10) & (8+9+R1): & s
\end{array}
$$

Each step above counts **0.5 pts.**

3. (3 pts.) Let $A = \{a, b\}$ and $B = \{\{a, b\}, c\}$. Determine:
   (a) $\mathcal{P}(A)$.
   (b) $A \cap B$.
   (c) $A \times B$.

Sol. Each part counts **1.0 pts.**

(a) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

(b) $A \cap B = \emptyset$. Note that $\{a, b\} \neq a$ and $\{a, b\} \neq b$.

(c)

$$
\begin{aligned}
A \times B &= \{a, b\} \times \{\{a, b\}, c\} \\
&= \{(a, \{a, b\}), (a, c), (b, \{a, b\}), (b, c)\}.
\end{aligned}
$$

The End.