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Tag : Toetsen/19-20/IntroMath-Calc1A.19-20[04].CorrectionModel.EN

Course : **Intro to Math + Calculus 1A**

Date : Friday November 8th, 2019

Time : 13:45 – 16:45

Solutions

1. [2 pt]

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2. [2 pt]

p	q	$(p \wedge \neg p) \leftrightarrow (p \to q)$
0	0	0
0	1	0
1	0	1
1	1	0

3. [3 pt] Case 1:

If x is even then $\exists n \in \mathbb{Z}\colon\ x = 2n$, hence

$$\begin{aligned} x^2 + 5x + 1 &= 4n^2 + 10n + 1 \\ &= 2\big(2n^2 + 5n\big) + 1, \end{aligned}$$

which is odd, since $2n^2 + 5n \in \mathbb{Z}$.

Case 2:

If x is odd then $\exists n \in \mathbb{Z}\colon\ x = 2n + 1$, hence

$$\begin{aligned} x^2 + 5x + 1 &= (2n+1)^2 + 5(2n+1) + 1 \\ &= 4n^2 + 14n + 7 \\ &= 2\big(2n^2 + 7n + 3\big) + 1, \end{aligned}$$

which is odd, since $2n^2 + 7n + 3 \in \mathbb{Z}$.

4. [4 pt] Let $S(k) : \displaystyle\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

Basis step for $k = 1$:

$\sum_{i=1}^{1} i^2 = 1^2 = 1$ which is equal to $\frac{1(1+1)(2\cdot 1+1)}{6} = 1$.
So the statement is correct for $k = 1$.

Induction step:
Suppose that $S(k)$ holds for some $k \in \mathbb{N}$:

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}. \text{ (Induction hypothesis: IH)}$$

We must show that the IH implies: $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}$.

Well: $\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$.

By IH this expression is equal to $\frac{k(k+1)(2k+1)}{6} + (k+1)^2$.
Finish the calculation:

$$\begin{aligned}\frac{k(k+1)(2k+1)}{6} + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{(k+1)\left(k(2k+1)+6(k+1)\right)}{6} \\ &= \frac{(k+1)\big(2k^2+7k+6\big)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}.\end{aligned}$$

Now we obtain from the principle of mathematical induction that $S(k)$ holds for all $k \in \mathbb{N}$.

5. (a) [1.5 pt]

$$\begin{aligned}\text{coefficient} &= \binom{9}{3} \\ &= \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} \\ &= 3\cdot 4\cdot 7 = 84.\end{aligned}$$

(b) [1.5 pt] The coefficient is $2^3 \cdot 3^6 \cdot \binom{9}{3}$.

6. (a) [1 pt] Calculate $\mathbf{u} = \overrightarrow{PQ} = \langle 0, 4, 4 \rangle$ and $\mathbf{v} = \overrightarrow{PR} = \langle -2, 0, 2 \rangle$, then

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 0 & 4 & 4 \\ -2 & 0 & 2 \end{bmatrix} \begin{matrix} 0 & 4 \\ -2 & 0 \end{matrix} = \boxed{\langle 8, -8, 8 \rangle}$$

Check Your Answer:

Verify that $\mathbf{u} \perp \mathbf{u} \times \mathbf{v}$ and that $\mathbf{v} \perp \mathbf{u} \times \mathbf{v}$.

(b) [2 pt] If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

Calculate the dot product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \cdot \mathbf{v} = 0 \cdot (-2) + 4 \cdot 0 + 4 \cdot 2 = 8.$$

Calculate the lengths of $\mathbf{u}$ and $\mathbf{v}$:

$$|\mathbf{u}| = \sqrt{0^2 + 4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}.$$

and

$$|\mathbf{v}| = \sqrt{(-2)^2 + 0^2 + 2^2} = 2\sqrt{2}.$$

Therefore

$$\cos\theta = \frac{8}{4\sqrt{2} \cdot 2\sqrt{2}} = \tfrac{1}{2},$$

and consequently $\theta = \boxed{\tfrac{1}{3}\pi}$ or $\theta = \boxed{60^\circ}$

Check Your Answer:

Use the property $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$ and the result of (a) to verify your answer.

(c) [2 pt] The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$(2(n+1)-1)3^{(n+1)+1} + 3\,\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}.$$

Caclulate the constituent parts:

$$\mathbf{u} \cdot \mathbf{v} = 8,$$

$$\mathbf{v} \cdot \mathbf{v} = 8.$$

Calculate the projection:

$$\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{8}{8}\mathbf{v} = \mathbf{v} = \boxed{\langle -2, 0, 2 \rangle}$$

Check Your Answer:
Verify that $\mathbf{v} \perp (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u})$.

(d) [2 pt] The normal equation of the plane V is

$$\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = 0,$$

with $\mathbf{n}$ a normal vector and $\mathbf{p}$ a support vector of V.
Using $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 8, -8, 8 \rangle$ as a normal vector and for example $\mathbf{p} = \overrightarrow{OP}$ as support vector we obtain the equation

$$\begin{aligned} \langle 8, -8, 8 \rangle \bullet \langle x - 1, y + 1, z + 1 \rangle &= 0, \\ 8x - 8y + 8z &= 8 \end{aligned}$$

which can be simplified to

$$\boxed{x - y + z = 1} \quad \text{or} \quad \boxed{z = 1 - x + y}$$

Check Your Answer:
Substitute the coordinates of P, Q and R in your equation, and verify whether it holds,

7. (a) [2 pt] Use the 'conjugate trick':

$$\begin{aligned} 1 - \sqrt{1 - x^2} &= 1 - \sqrt{1 - x^2} \cdot \frac{1 + \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \\ &= \frac{1 - (1 - x^2)}{1 + \sqrt{1 - x^2}} = \frac{x^2}{1 + \sqrt{1 - x^2}} \end{aligned}$$

Alternatively, you can prove that $\left(1 - \sqrt{1 - x^2}\right)\left(1 + \sqrt{1 - x^2}\right) = x^2$:

$$\begin{aligned} \left(1 - \sqrt{1 - x^2}\right)\left(1 + \sqrt{1 - x^2}\right) &= 1 - \left(\sqrt{1 - x^2}\right)^2. \\ &= 1 - \left(1 - x^2\right) = x^2. \end{aligned}$$

(b) [2 pt] Use (a) to rewrite:

$$\frac{\sqrt{1-\sqrt{1-x^2}}}{x} = \frac{1}{x}\sqrt{\frac{x^2}{1+\sqrt{1-x^2}}} = \frac{|x|}{x}\sqrt{\frac{1}{1+\sqrt{1-x^2}}}$$

$$= \begin{cases} +\sqrt{\dfrac{1}{1+\sqrt{1-x^2}}} & \text{if } x > 0, \\ -\sqrt{\dfrac{1}{1+\sqrt{1-x^2}}} & \text{if } x < 0. \end{cases}$$

Now use $\lim_{x\to 0}\sqrt{\frac{1}{1+\sqrt{1-x^2}}} = \sqrt{\frac{1}{1+\sqrt{1-0^2}}} = \frac{1}{2}\sqrt{2}$ to conclude:

$$\lim_{x\to 0^+}\frac{\sqrt{1-\sqrt{1-x^2}}}{x} = \tfrac{1}{2}\sqrt{2} \quad \text{and} \quad \lim_{x\to 0^-}\frac{\sqrt{1-\sqrt{1-x^2}}}{x} = -\tfrac{1}{2}\sqrt{2}.$$

Since left- and right limit are not equal, the two-sided limit of $\frac{\sqrt{1-\sqrt{1-x^2}}}{x}$ for $x \to 0$ does not exist.

(c) [2 pt] The derivative of f at 0 is

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{\sqrt{1-\sqrt{1-h^2}}}{h},$$

which does not exist according to assignment (b).

8. [3 pt] Differentiate f:

$$f'(x) = -(x^2+2x-3)e^x.$$

Since f is a differentiable function, the only critical points are points x where $f'(x) = 0$.

This is the case whenever $x^2+2x-3 = (x+3)(x-1) = 0$, so if $x = -3$ or $x = 1$. The only critical point of f in $\left[-1, \sqrt{3}\right]$ is 1.

Other candidates for extreme values are the boundaries of $\left[-1, \sqrt{3}\right]$.

x	$f(x)$
-1	$\frac{2}{e}$
1	$2e$
$\sqrt{3}$	0

From the table we conclude: the absolute minimum value is 0 and the absolute maximum value is $2e$.

9. [3 pt] Use polar coordinates:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} &= \lim_{r\to 0^+} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \lim_{r\to 0^+} r\frac{\cos^3\theta+\sin^3\theta}{\cos^2\theta+\sin^2\theta} \\ &= \lim_{r\to 0^+} r\big(\cos^3\theta+\sin^3\theta\big).\end{aligned}$$

Observe that $\cos^3\theta+\sin^3\theta$ is bounded.

Since $\lim\limits_{r\to 0^+} r = 0$, the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

10. [3 pt] The equation for the tangent plane through $\big(a, b, f(a,b)\big)$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\big(x-a\big) + \frac{\partial f}{\partial y}(a,b)\big(y-b\big). \qquad (*)$$

Calculate the partial derivatives of f:

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x+2y-2},$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2}{x+2y-2}.$$

Evaluate f and the partial derivatives at $(a,b)=(1,1)$:

$$f\big(1,1\big) = \ln 1 = 0,$$

$$\frac{\partial f}{\partial x}(1,1) = 1, \quad \frac{\partial f}{\partial y}(1,1) = 2.$$

Write down the equation of the tangent plane (fill out all results in $(*)$):

$$\begin{aligned}z &= 0 + 1(x-1) + 2(y-1), \\ z &= x + 2y - 3.\end{aligned}$$

The equation may be rearranged, like

$$x + 2y - z = 3.$$

Total: 36 points