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Tag : Toetsen/19-20/Calc1A.19-20[03].CorrectionModel.EN

Course : **Calculus 1A**

Date : Friday October 25th, 2019

Time : 13:45 – 15:45

Solutions

1. (a) [1 pt] Calculate $\mathbf{u} = \overrightarrow{PQ} = \langle 2, 2, -1 \rangle$ and $\mathbf{v} = \overrightarrow{PR} = \langle 3, 0, -3 \rangle$, then

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 2 & 2 & -1 \\ 3 & 0 & -3 \end{bmatrix} \begin{matrix} 2 & 2 \\ 3 & 0 \end{matrix} = \boxed{\langle -6, 3, -6 \rangle}$$

Note: Round brackets are also acceptable.

Check Your Answer:
Verify that $\mathbf{u} \perp \mathbf{u} \times \mathbf{v}$ and that $\mathbf{v} \perp \mathbf{u} \times \mathbf{v}$.

(b) [2 pt] If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

Calculate the dot product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \bullet \mathbf{v} = 2 \cdot 3 + 2 \cdot 0 + (-1) \cdot (-3) = 9.$$

Calculate the lengths of $\mathbf{u}$ and $\mathbf{v}$:

$$|\mathbf{u}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

and

$$|\mathbf{v}| = \sqrt{3^2 + 0^2 + (-3)^2} = 3\sqrt{2}.$$

Therefore

$$\cos\theta = \frac{9}{3 \cdot 3\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

and consequently $\theta = \boxed{\tfrac{1}{4}\pi}$ or $\theta = \boxed{45\,^\circ}$

Check Your Answer:
Use the property $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$ and the result of (a) to verify your answer.

(c) [2 pt] The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v}.$$

Caclulate the constituent parts:

$$\mathbf{u} \cdot \mathbf{v} = 9,$$
$$\mathbf{v} \cdot \mathbf{v} = 18.$$

Calculate the projection:

$$\text{proj}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{9}{18}\mathbf{v} = \boxed{\left\langle \tfrac{3}{2}, 0, -\tfrac{3}{2} \right\rangle}$$

Note: Round brackets are also acceptable.

Check Your Answer:
Verify that $\mathbf{v} \perp (\mathbf{u} - \text{proj}_{\mathbf{v}}\, \mathbf{u})$.

(d) [2 pt] The normal equation of the plane V is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0,$$

with $\mathbf{n}$ a normal vector and $\mathbf{p}$ a support vector of V.
Using $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle -6, 3, -6 \rangle$ as a normal vector and for example $\mathbf{p} = \overrightarrow{OP}$ as support vector we obtain the equation

$$\begin{aligned} \langle -6, 3, -6 \rangle \cdot \langle x+1, y-1, z-2 \rangle &= 0, \\ -6x + 3y - 6z &= -3 \end{aligned}$$

which can be simplified to

$$\boxed{2x - y + 2z = 1} \quad \text{or} \quad \boxed{z = -x + \tfrac{1}{2}y + \tfrac{1}{2}}$$

Check Your Answer:
Substitute the coordinates of p, Q and R in your equation, and verify whether it holds,

2. (a) [1 pt] Use the 'conjugate trick':

$$\begin{aligned} \sqrt{1 - \cos x} &= \sqrt{1 - \cos x} \cdot \frac{\sqrt{1 + \cos x}}{\sqrt{1 + \cos x}} \\ &= \frac{\sqrt{1 - \cos^2 x}}{\sqrt{1 + \cos x}} = \frac{\sqrt{\sin^2 x}}{\sqrt{1 + \cos x}} \end{aligned}$$

Alternatively, you can prove that $\sqrt{1 - \cos x}\sqrt{1 + \cos x} = \sqrt{\sin^2 x}$:

$$\begin{aligned} \sqrt{1 - \cos x}\sqrt{1 + \cos x} &= \sqrt{(1 - \cos x)(1 + \cos x)} \\ &= \sqrt{1 - \cos^2 x} \\ &= \sqrt{\sin^2 x} \end{aligned}$$

(b) [2 pt] Use (a) to rewrite:

$$\frac{\sqrt{1-\cos x}}{x} = \frac{\sqrt{\sin^2 x}}{x\sqrt{1+\cos x}}$$

$$= \frac{|\sin x|}{x\sqrt{1+\cos x}}$$

$$= \begin{cases} \dfrac{\sin x}{x} \cdot \dfrac{1}{\sqrt{1+\cos x}} & \text{if } x > 0, \\ -\dfrac{\sin x}{x} \cdot \dfrac{1}{\sqrt{1+\cos x}} & \text{if } x < 0. \end{cases}$$

Now use $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1}{\sqrt{1+\cos x}} = \frac{1}{\sqrt{1+1}} = \frac{1}{2}\sqrt{2}$ to conclude:

$$\lim_{x \to 0^+} \frac{\sqrt{1-\cos x}}{x} = \tfrac{1}{2}\sqrt{2} \quad \text{and} \quad \lim_{x \to 0^-} \frac{\sqrt{1-\cos x}}{x} = -\tfrac{1}{2}\sqrt{2}.$$

Since left- and right limit are not equal, the two-sided limit of $\frac{\sqrt{1-\cos x}}{x}$ for $x \to 0$ does not exist.

Note: do not award any points for this assignment if $\sin x$ is used instead of $|\sin x|$. Using $\sin x$ would lead to the conclusion that the limit does exist, which clearly contradicts the assumption in the assignment text.

(c) [2 pt] The derivative of f at 0 is

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\sqrt{1-\cos h} - 0}{h},$$

which does not exist according to assignment (b).

(d) [2 pt] Differentiate f:

$$f'(x) = \frac{\sin x}{2\sqrt{1-\cos x}}$$

So $f'(x) = 0$ whenever $\sin x = 0$, but $\cos x \neq 1$. For x in the interval $\left[-\frac{1}{2}\pi, \frac{3}{2}\pi\right]$, this is only the case whenever $\boxed{x = \pi}$

If the answer is π:

If the answer is $0, \pi$:

If the answer is $n\pi$ with n odd:

All other answers:

(e) [1 pt] The absolute minimum is $\boxed{0}$

(f) [1 pt] The absolute maximum is $\boxed{\sqrt{2}}$

Calculation:

Critical points of f on $D = \left[-\frac{1}{2}\pi, \frac{3}{2}\pi\right]$ are 0 and π. Other candidates for extreme values are the boundaries of D.

x	$f(x)$
$-\frac{1}{2}\pi$	1
0	0
π	$\sqrt{2}$
$\frac{3}{2}\pi$	1

3. [3 pt] If polar coordinates are used, the calculation should look like this:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^4+y^4}} = \lim_{r\to 0^+} \frac{r^2\cos\theta\sin\theta}{\sqrt{r^4(\cos^4\theta+\sin^4\theta)}} = \lim_{r\to 0^+} \frac{\cos\theta\sin\theta}{\sqrt{\cos^4\theta+\sin^4\theta}},$$

which depends on θ. So for example if $\theta = 0$ (approach $(0,0)$ along the positive x-axis), then the limit is 0 (because $\cos\theta = 1$ and $\sin\theta = 0$). But if for example $\theta = \frac{1}{4}\pi$ (approach $(0,0)$ along the positive y-axis), then the limit is $\frac{1}{2}\sqrt{2}$.

If approaching $(0,0)$ along a line is used: let $y = \alpha x$ for some value of α, then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along line } y=\alpha x}} \frac{xy}{\sqrt{x^4+y^4}} = \lim_{x\to 0} \frac{\alpha x^2}{\sqrt{(1+\alpha^4)x^4}} = \frac{\alpha}{\sqrt{1+\alpha^4}}.$$

So for instance, if (x,y) is on the x-axis, then $y = 0$, which can be achieved by choosing $\alpha = 0$. In that case the limit is 0.

If (x,y) is on the line $y = x$, then $\alpha = 1$, and consequently the limit is $\frac{1}{2}\sqrt{2}$.

4. [3 pt] The equation for the tangent plane through $\big(a, b, f(a,b)\big)$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\big(x-a\big) + \frac{\partial f}{\partial y}(a,b)\big(y-b\big). \qquad (*)$$

Calculate the partial derivatives of f:

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 - y,$$

$$\frac{\partial f}{\partial y}(x,y) = -x - 2y.$$

Evaluate f and the partial derivatives at $(a,b) = (1,-1)$:

$$f\big(1,-1\big) = 1,$$

$$\frac{\partial f}{\partial x}(1,-1) = 4,$$

$$\frac{\partial f}{\partial y}(1,-1) = 1.$$

Write down the equation of the tangent plane (fill out all results in $(*)$):

$$z = 1 + 4(x-1) + 1\big(y-(-1)\big),$$
$$z = 4x + y - 2.$$

The equation may be rearranged, like

$$4x + y - z = 2.$$

Total: 22 points