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Course : **Calculus 1A**

Solutions

1. (a) [1 pt] Define $\mathbf{u} = \langle -1, 2, 2\rangle$ and $\mathbf{v} = \langle 4, -3, 0\rangle$, then

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} -1 & 2 & 2 \\ 4 & -3 & 0 \end{bmatrix} \begin{matrix} -1 & 2 \\ 4 & -3 \end{matrix} = \boxed{\langle 6, 8, -5\rangle}$$

Check Your Answer:
Check that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

(b) [2 pt] If θ is the angle at P, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

Calculate the dot product of $\mathbf{u}$ and $\mathbf{v}$, and the lengths of $\mathbf{u}$ and $\mathbf{v}$:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= -10, \\ |\mathbf{u}|^2 &= \mathbf{u} \cdot \mathbf{u} = 9, \\ |\mathbf{v}|^2 &= \mathbf{v} \cdot \mathbf{v} = 25. \end{aligned}$$

Therefore $\cos\theta = \dfrac{-10}{3 \cdot 5} = -\frac{2}{3}$.

Check Your Answer:
Observe that $\sin^2\theta = 1 - \cos^2\theta = \frac{5}{9}$. Use this to check that the equality $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ holds.

(c) [2 pt] The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$$

With $\mathbf{u} \cdot \mathbf{v} = -10$ and $|\mathbf{v}|^2 = 25$ (see (c)), we obtain

$$\begin{aligned} \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= -\frac{10}{25}\mathbf{v} = -\frac{2}{5}\mathbf{v} \\ &= \boxed{\left\langle -\frac{8}{5}, \frac{6}{5}, 0 \right\rangle} \end{aligned}$$

Check Your Answer:
With $\mathbf{w} = \operatorname{proj}_{\mathbf{v}} \mathbf{u}$, check that $(\mathbf{u} - \mathbf{w}) \perp \mathbf{v}$.

2. [3 pt] The normal equation of the plane is

$$\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = 0,$$

with $\mathbf{n}$ a normal vector and $\mathbf{p}$ a support vector of V.

Using $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ as a normal vector and for example $\mathbf{p} = \overrightarrow{OP}$ as support vector we obtain the equation

$$\langle -9, 3, -3 \rangle \bullet \langle x-3, y-2, z+1 \rangle = 0,$$

) divide by 3

$$\langle -3, 1, -1 \rangle \bullet \langle x-3, y-2, z+1 \rangle = 0,$$

) simplify the dot product

which can be simplified to

$$\boxed{3x - y + z = 6} \quad \text{or} \quad \boxed{z = 6 - 3x + y}$$

Check Your Answer:

For x, y and z, fill in the coordinates of P, and check whether the equation holds. Do the same for Q and R.

3. (a) [3 pt] **Method 1: use L'Hôpital**

Note that the limit is of type "$\frac{0}{0}$", so the use of L'Hôpitals rule is justified.

Calculate the limit using L'Hôpital:

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1} = \lim_{x \to 0} \frac{1}{\frac{1}{2\sqrt{x+1}} - 0}. = \lim_{x \to 0} 2\sqrt{x+1} = 2.$$

Method 2: use the conjugate trick

$$\begin{aligned}
\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1} &= \lim_{x \to 0} \frac{x(\sqrt{x+1} + 1)}{(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)} \\
&= \lim_{x \to 0} \frac{x(\sqrt{x+1} + 1)}{(\sqrt{x+1})^2 - 1^2} \\
&= \lim_{x \to 0} \frac{x(\sqrt{x+1} + 1)}{(x + \cancel{1}) - \cancel{1}} \\
&= \lim_{x \to 0} \frac{\cancel{x}(\sqrt{x+1} + 1)}{\cancel{x}} \\
&= \sqrt{0+1} + 1 = 2.
\end{aligned}$$

(b) [2 pt] Left and right limit for $x \to 0$ of f must be equal, so we require

$$\lim_{x \to 0^-} f(x) = 2.$$

This gives $-p = 2$, hence $\boxed{p = -2}$

4. (a) [2 pt] The function f is a polynomial, so the only critical points are values x for which $f'(x) = 0$. The derivative of f is

$$f'(x) = 3x^2 - 2x - 1.$$

Solving the equation $f'(x) = 0$ gives $x = -\frac{1}{3}$ and $x = 1$, so these are critical points.

(b) [1 pt] Candidates for the extreme values of f on $[-2, 2]$ are the boundaries -2 and 2, as well as the critical points found in (a), which are in the interval $(-2, 2)$.

x	$f(x)$
-2	-9
$-\frac{1}{3}$	$\frac{32}{27}$
1	0
2	3

The absolute minimum is -9, and the absolute maximum is 3, so the correct answer is answer 1.

5. (a) [3 pt] If polar coordinates are used, the calculation would look like this:

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2} = \lim_{r\to 0^+} \frac{r\cos\theta - r\sin\theta}{r^2} = \lim_{r\to 0^+} \frac{\cos\theta - \sin\theta}{r},$$

which does not exist. So for example if $\theta = 0$ (approach $(0,0)$ along the positive x-axis), then $y = 0$, hence

$$\lim_{x\to 0^+} \frac{x-0}{x^2+0^2} = \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

(b) [3 pt] The equation for the tangent plane V through $\big(a, b, c\big)$ with $c = f(a, b)$ is

$$z = c + \frac{\partial f}{\partial x}(a,b)\big(x-a\big) + \frac{\partial f}{\partial y}(a,b)\big(y-b\big). \qquad (*)$$

Calculate the partial derivatives of f:

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^2 + 2xy - x^2}{\big(x^2+y^2\big)^2},$$
$$\frac{\partial f}{\partial y}(x,y) = \frac{y^2 - 2xy - x^2}{\big(x^2+y^2\big)^2}.$$

Evaluate f and the partial derivatives at $(a, b) = (1, -1)$:

$$c = f\big(1, -1\big) = 1,$$

$$\frac{\partial f}{\partial x}(1, -1) = -\tfrac{1}{2},$$

$$\frac{\partial f}{\partial y}(1, -1) = \tfrac{1}{2}.$$

Write down the equation of V (fill out all results in $(*)$):

$$z = 1 - \frac{1}{2}(x - 1) + \tfrac{1}{2}\big(y - (-1)\big),$$

$$z = 2 - \tfrac{1}{2}x + \tfrac{1}{2}y.$$

The equation may be rearranged, like

$$2z = 4 - x + y,$$

$$\text{or:} \quad x - y + 2z = 4.$$

Check Your Answer:
Check whether $x = 1$, $y = -1$, $z = 1$ satisfies the equation.

Total: 22 points