Questions

1	2	3	4	5	6	7	8	9	10
11	12								

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

Surname, First name

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Calculus 1B
Calculus 1B test (A31)
10 January 2020 13:45 - 15:45

Study programme:

- Write your **student number** in the top right section, colouring in the correct number sequence.
- Use a black or blue pen. Do not use a pencil.
- The use of a calculator or any other electronic device is not allowed.
- Please turn off your cell phone.

- This exam consists of 11 questions.
- The last page is extra writing space. Clearly refer to it if you make use of it.

- This exam consists of a total of 36 points:
- Multiple Choice (8 points): Q1, Q5 and Q9.
- Final Answer (10 points): Q2, Q3, Q7 and Q8.
- Open (18 points): Q4, Q6, Q10 and Q11.

- **Do not remove this page**

Question 1

2p The expression $\sum_{k=1}^{n} \frac{2}{n(1+\frac{2k}{n})^2}$ is a Riemann sum for a function on an interval. Decide which integral equals the limit of this expression if n tends to ∞.

- A) $\int_0^1 \frac{2}{(1-2x)^2}dx$
- B) $\int_0^2 \frac{2}{(1+2x)^2}dx$
- C) $\int_0^2 \frac{1}{(1+x)^2}dx$
- D) $\int_0^1 \frac{2}{(1-x)^2}dx$
- E) $\int_0^2 \frac{1}{(1-x)^2}dx$
- F) $\int_0^2 \frac{1}{(1+2x)^2}dx$
- G) $\int_0^1 \frac{1}{(1+2x)^2}dx$
- H) $\int_0^1 (\frac{2}{1+2x})^2dx$

Question 2

Only write your final answer to the question in the box below.

3p Determine $\frac{dy}{dx}$ in case $y = \int_x^{x^3} e^{t^2}\sin(t)dt$

$\frac{dy}{dx} =$ $3x^2 e^{x^6} \sin(x^3) - e^{x^2}\sin(x)$

Question 3

Only write your final answer to the question in the box below.

2p

Determine $\int \cos(x)\sqrt[5]{2+\sin(x)}dx =$ $\frac{5}{6}(2+\sin(x))^{\frac{6}{5}} + C$

Question 4

5p Compute $\int_0^\infty \frac{x}{e^{2x}}dx$

$$\int_0^\infty \frac{x}{e^{2x}}dx = \lim_{b\to\infty} \int_0^b \frac{x}{e^{2x}}dx$$

First, $\int \frac{x}{e^{2x}}dx = \int x e^{-2x}dx = \int x\, d(-\frac{1}{2}e^{-2x}) =$

$x \cdot -\frac{1}{2}e^{-2x} - \int -\frac{1}{2}e^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx =$

$-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$. Therefore

$$\int_0^b \frac{x}{e^{2x}}dx = -\frac{1}{2}be^{-2b} - \frac{1}{4}e^{-2b} - (0 - \frac{1}{4})$$

Furthermore, $\lim_{b\to\infty} -\frac{1}{2}be^{-2b} = 0$ and $\lim_{b\to\infty} e^{-2b} = 0$

(remark $\lim_{b\to\infty}\frac{b}{e^{2b}} = $ L'Hopital $\lim_{b\to\infty}\frac{1}{2e^{2b}} = 0$)

Conclusion:

$$\int_0^\infty \frac{x}{e^{2x}}dx = \frac{1}{4}$$

Question 5

3p Compute $\sum_{n=0}^{\infty} \frac{2^n + 3^n}{6^n}$

- ○ A) 6
- ○ B) $\frac{5}{6}$
- ● C) $\frac{7}{2}$
- ○ D) 4
- ○ E) $\frac{9}{2}$
- ○ F) $\frac{8}{3}$
- ○ G) $\frac{6}{5}$
- ○ H) 3

Question 6

3p Find the Taylor polynomial of order 4 generated by $f(x) = \frac{1}{x}$ at $a = 2$

$f(x) = \frac{1}{x} = x^{-1}$ $\qquad f(2) = \frac{1}{2}$

$f'(x) = -1x^{-2} = \frac{-1}{x^2}$ $\qquad f'(2) = -\frac{1}{4}$

$f''(x) = -1 \cdot -2x^{-3} = \frac{2}{x^3}$ $\qquad f''(2) = \frac{2}{8} = \frac{1}{4}$

$f'''(x) = -1 \cdot -2 \cdot -3x^{-4} = \frac{-6}{x^4}$ $\qquad f'''(2) = \frac{-6}{16} = -\frac{3}{8}$

$f^{(iv)}(x) = -1 \cdot -2 \cdot -3 \cdot -4x^{-5} = \frac{24}{x^5}$ $\qquad f^{(iv)}(2) = \frac{24}{32} = \frac{3}{4}$

The Taylor polynomial of order 4 is denoted by $P_4(x)$

$$P_4(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{iv}(2)}{4!}(x-2)^4$$

$$P_4(x) = \frac{1}{2} + \left(-\frac{1}{4}\right)(x-2) + \frac{1/4}{2!}(x-2)^2 + \frac{-3/8}{3!}(x-2)^3 + \frac{3/8}{4!}(x-2)^4$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4$$

Question 7

Only write your final answer to the question in the box below.

3p Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dx} + \dfrac{3}{x}y = \dfrac{\sin(x)}{x^2} \\ y\left(\dfrac{\pi}{2}\right) = 1 \end{cases}$$

$y =$ $\dfrac{-\cos(x)}{x^2} + \dfrac{\sin(x)}{x^3} + \dfrac{\left(\frac{\pi}{2}\right)^3 - 1}{x^3}$

Question 8

Only write your final answer to the question in the box below.

2p A population $P(t)$ is modeled by the initial value problem

$$\begin{cases} \dfrac{dP}{dt} = 1.2P\left(1 - \dfrac{P}{4200}\right). \\ P(0) = c. \end{cases}$$

For which values of c is the population increasing?

$0 < c < 4200$

Question 9

3p Given $z_1 = 3e^{\frac{\pi i}{4}}$ and $z_2 = 4e^{\frac{-\pi i}{6}}$. Find the Cartesian form for $z_1 z_2$.
Choose from the alternatives below.

- ○ A) $12\sqrt{2} + 12i\sqrt{3}$
- ○ B) $12\sqrt{2} - 12i\sqrt{3}$
- ○ C) $(2\sqrt{6} + 2\sqrt{2}) + i(2\sqrt{6} + 2\sqrt{2})$
- ○ D) $(2\sqrt{6} + 2\sqrt{2}) - i(2\sqrt{6} + 2\sqrt{2})$
- ○ E) $(3\sqrt{6} - 3\sqrt{2}) + i(3\sqrt{6} + 3\sqrt{2})$
- ● F) $(3\sqrt{6} + 3\sqrt{2}) + i(3\sqrt{6} - 3\sqrt{2})$
- ○ G) $12\sqrt{3} + 12i\sqrt{2}$
- ○ H) $12\sqrt{3} - 12i\sqrt{2}$

Question 10

3p Find all solutions in $\mathbb{C}$ of the equation $z^3 = i$

Write $z = re^{i\theta}$ and $i = 1e^{i\frac{\pi}{2}}$ in polar form

$z^3 = i \iff r^3 e^{i3\theta} = 1e^{i\frac{\pi}{2}} \iff \begin{cases} r^3 = 1 \\ 3\theta = \frac{\pi}{2} + k\cdot 2\pi \quad k \in \mathbb{Z} \end{cases}$

It follows that $r = 1$ and $\theta = \frac{\pi}{6} + k\cdot\frac{2}{3}\pi$ with $k \in \mathbb{Z}$

Solutions: $z_1 = 1e^{i\frac{\pi}{6}} = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$

$z_2 = 1e^{i\frac{5}{6}\pi} = -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

$z_3 = 1e^{i\frac{9}{6}\pi} = -i$

Question 11

7p Determine the unique solution to the problem

$$\begin{cases} y'' + 6y' + 5y = e^{-t} \\ y(0) = 0 \\ y'(0) = 3 \end{cases}$$

At first we solve $y'' + 6y' + 5y = 0$ (homogeneous eq.)

$r^2 + 6r + 5 = 0$ (characteristic equation)

$\Leftrightarrow$ $r_1 = -5$ and $r_2 = -1$. We've found

$y_H(t) = c_1 e^{-5t} + c_2 e^{-t}$ with $c_1, c_2 \in \mathbb{R}$

Secondly, we try a particular solution for inhomogeneous equation, $y_p(t) = Ate^{-t}$ (remark $y = c_2 e^{-t}$ is solution of the homogeneous equation, therefore Ate^{-t})

$y_p'(t) = Ae^{-t} - Ate^{-t}$

$y_p''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$

Substituting this in $y'' + 6y' + 5y = e^{-t}$ results

$-2Ae^{-t} + Ate^{-t} + 6(Ae^{-t} - Ate^{-t}) + 5Ate^{-t} = e^{-t}$

So

$$(-2A + At + 6A - 6At + 5At)\, e^{-t} = e^{-t}$$

or $4A\, e^{-t} = e^{-t} \Rightarrow A = \frac{1}{4}$, $y_p(t) = \frac{1}{4} t e^{-t}$

Summarizing, the general solution of the inhom. eq. is

$$y(t) = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{4} t e^{-t} = y_H(t) + y_p(t)$$

with $y'(t) = -5c_1 e^{-5t} - c_2 e^{-t} + \frac{1}{4} e^{-t} - \frac{1}{4} t e^{-t}$

Taking into account the initial conditions

$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$y'(0) = 3 \Rightarrow -5c_1 - c_2 + \frac{1}{4} = 3$

It follows: $-5c_1 + c_1 + \frac{1}{4} = 3 \Rightarrow -4c_1 = \frac{11}{4} \Rightarrow c_1 = -\frac{11}{16}$

and $c_2 = +\frac{11}{16}$

Conclusion: the unique solution is given by

$$y(t) = -\frac{11}{16} e^{-5t} + \frac{11}{16} e^{-t} + \frac{1}{4} t e^{-t}$$