Course : **Calculus 1 B**

Date : January 11, 2019

Time : 13.45 - 15.45

Solution Set

[2p] **1.** Splitting the integral: $y = \int_1^{3x} \ln(t)\,dt - \int_1^{2x} \ln(t)\,dt$

Use Fund. Th., Chain rule: $\dfrac{dy}{dx} = 3\ln(3x) - 2\ln(2x)$

or

An anti derivative of $\ln(t)$ is $t\ln(t) - t$

Then $y = 3x\ln(x) - 3x - (2x\ln(2x) - 2x)$

$$\frac{dy}{dx} = 3\ln(3x) + \frac{3x}{3x}\cdot 3 - 3 - \left(2\ln(2x) + \frac{2x}{2x}\cdot 2 - 2\right)$$

Answer: $3\ln(3x) - 2\ln(2x)$

[4p] **2.** Choice of $f(x) = (1+x)^4$ with $0 \le x \le 1$ and region

The given sum is a Riemann sum (upper/right hand endpoint) of $f(x)$

belonging to division of $[0,1]$ in n equal subintervals

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \frac{k}{n}\right)^4 \cdot \frac{1}{n} = \int_0^1 (1+x)^4\,dx$$

$$\int_0^1 (1+x)^4\,dx = \left[\tfrac{1}{5}(1+x)^5\right]_{x=0}^{1} = \frac{31}{5}$$

[2p] **3**. a) Substitute $u = \dfrac{1}{x}$

Transformed integral: $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx = \int -e^u du$

$\displaystyle\int -e^u du = -e^u$, so $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx = -e^{\frac{1}{x}}$

$$\int_{x=1}^{2} \frac{e^{\frac{1}{x}}}{x^2}\,dx = \left[-e^{\frac{1}{x}}\right]_{x=1}^{2} = e - \sqrt{e}$$

[3p] b) Part. Int.: $\displaystyle\int 2x\tan^{-1}(x)\,dx = \left[x^2\tan^{-1}(x)\right] - \int x^2\,d\tan^{-1}(x)$

$$\int x^2\,d\tan^{-1}(x) = \int \frac{x^2}{x^2+1}\,dx$$

$$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = x - \tan^{-1}(x)$$

Answer: $x^2\tan^{-1}(x) - x + \tan^{-1}(x) + C$

[3p] c) $\displaystyle\int_e^{\infty} \frac{1}{x(\ln(x))^3}\,dx = \lim_{b\to\infty}\int_e^{b} \frac{1}{x(\ln(x))^3}\,dx$

$\displaystyle\int \frac{1}{x(\ln(x))^3}\,dx =$ (via subst. $u = \ln(x)$) $= -\tfrac{1}{2}(\ln(x))^{-2}$

$$\int_e^{b} \frac{1}{x(\ln(x))^3}\,dx = -\tfrac{1}{2}(\ln(b))^{-2} + \tfrac{1}{2}(\ln(e))^{-2}$$

$$\lim_{b\to\infty}\left(-\tfrac{1}{2}(\ln(b))^{-2} + \tfrac{1}{2}(\ln(e))^{-2}\right) = 0 + \frac{1}{2} = \tfrac{1}{2}$$

[2p] **4.** a) The given series is a geometric series with ratio $\frac{x}{2}$, therefore convergent in case $-1 < \frac{x}{2} < 1$, so $-2 < x < 2$

The sum of the series $= \dfrac{\text{first term}}{1 - \text{ratio}} = \dfrac{\frac{x}{2}}{1 - \frac{x}{2}} \qquad \left(= \dfrac{x}{2-x}\right)$

[2p] b) The given series is the derivative of the series in **4**.a), thus

$$\sum_{n=1}^{\infty} \frac{n\,x^{n-1}}{2^n} = \frac{d}{dx}\left(\frac{\frac{x}{2}}{1-\frac{x}{2}}\right) = \frac{\frac{1}{2}}{(1-\frac{x}{2})^2} \qquad \left(= \frac{2}{(2-x)^2}\right)$$

[4p] **5.** Solution by separating: $\dfrac{1}{y}\,dy = \dfrac{x}{x^2+1}\,dx$

Integrate both sides: $\ln|y| = \frac{1}{2}\ln(x^2+1) + C$

Then $y = k\sqrt{x^2+1}$ with k a constant

$y(0) = 1 \Rightarrow k = 1$, answer: $y(x) = \sqrt{x^2+1}$

[$\ln(y)$ in stead of $\ln|y|$ and $k > 0$ as a consequence: $-\frac{1}{2}$ p]

[2p] **6.** a) Using polar coordinates $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

Therefore $(1+i)^{200} = 2^{100}\,e^{i50\pi} = 2^{100}$ is the real (part)

or

$(1+i)^2 = 2i$ and so $(1+i)^4 = -4$

Therefore $(1+i)^{200} = (-4)^{50} = 2^{100}$ is the real (part)

[2p] b) $z^4 + z^3 + z^2 = z^2(z^2+z+1) = 0 \Leftrightarrow$

$z^2 = 0$ or $z^2 + z + 1 = 0 \Leftrightarrow$

$z = 0$ (double root) or $z = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$ or $z = -\frac{1}{2} - \frac{1}{2}i\sqrt{3}$

[2p] c) Polar form $w_1 = 2\sqrt{3} - 2i = 4\,e^{-i\frac{\pi}{6}}$

Polar form $w_2 = -1 + i = \sqrt{2}\,e^{i\frac{3}{4}\pi}$

Polar form $\dfrac{w_1}{w_2} = \dfrac{4}{\sqrt{2}}\,e^{-i\frac{\pi}{6} - i\frac{3}{4}\pi} = 2\sqrt{2}\,e^{-i\frac{11}{12}\pi}$

[6p] **7.** Solve the homogeneous equation: $y'' - 2y' + y = 0$.

Its characteristic equation is $r^2 - 2r + 1 = 0 \Leftrightarrow r = 1$ (double root)

So $y_h(x) = c_1\, e^x + c_2\, x\, e^x$ (with $c_1, c_2 \in \mathbb{R}$)

(Try) the particular solution $y_p(x) = Ax + B$, then

$y_p' = A$, $y_p'' = 0$, so $\forall x : 0 - 2A + Ax + B = x \Rightarrow A = 1, B = 2$

The general solution is $y(x) = x + 2 + c_1\, e^x + c_2\, x\, e^x$ with

$y'(x) = 1 + c_1\, e^x + c_2\, e^x + c_2\, x\, e^x$

$y(0) = 1 \Rightarrow 2 + c_1 = 1 \Rightarrow c_1 = -1$

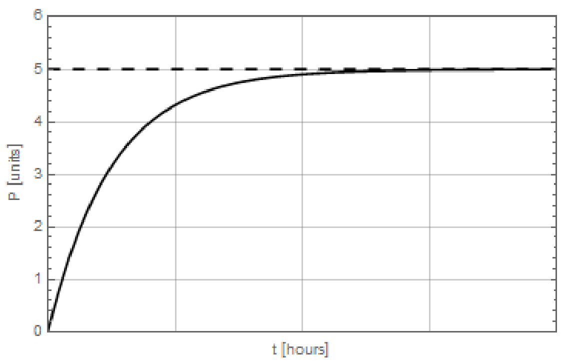
$y'(0) = 1 \Rightarrow 1 + c_1 + c_2 = 1 \Rightarrow c_2 = 1$

Answer: $y(x) = x + 2 - e^x + x\, e^x$

[2p] **8.** A sketch shows an increasing $P(t)$ starting at $P(0) = 0$

and bounded (above) by $P_{max} = 5$

Possible graph of $P(t)$ (sketch, including names of variables at axes and $P_{max} = 5$ shown), e.g.:



The End.

Total: 36 points