1. a)

$$\int_{-x}^{0} \sin\left(t^2\right) dt = -\int_{0}^{-x} \sin\left(t^2\right) dt$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_{0}^{-x} \sin\left(t^2\right) dt\right) = -\sin\left(-x\right)^2 \cdot -1$$

Answer: $\sin\left(x^2\right)$

b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int_{0}^{x} e^{-t^2} dt + x \cdot e^{-x^2}$$

2. a)

Substitute $\ln(x) = u$

Then $du = \frac{1}{x} dx$

Transform integral $\int \frac{1}{\sqrt{u}} du$

$$\int \frac{1}{\sqrt{u}} du = 2\sqrt{u}$$

$$\int_{e}^{e^2} \frac{1}{x\sqrt{\ln(x)}} dx = \left[2\sqrt{\ln(x)}\right]_{e}^{e^2} = 2\sqrt{\ln\left(e^2\right)} - 2\sqrt{\ln(e)}$$

$$= 2\sqrt{2} - 2$$

b)

Use partial integration twice:

$$\int x^2 e^{-x} dx = \int x^2 d(-e^{-x}) = -x^2 e^{-x} - \int -e^{-x} dx^2$$

$$\int e^{-x} dx^2 = \int 2xe^{-x} dx = 2\left(-xe^{-x} - e^{-x}\right)$$

Answer: $-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$

c)

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}$$

$$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}$$

$$\lim_{b\to\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} - -\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

3. **a)**

Series is a geometrical one with ratio $(\ln(x)+1)$

Series converges $\iff |\ln(x)+1| < 1$, so $-1 < \ln(x)+1 < 1$

$-1 < \ln(x)+1 < 1 \iff e^{-2} < x < 1$, answer $\left(e^{-2}, 1\right)$

Series sum $= \dfrac{\text{first term}}{1-\text{ratio}} = \dfrac{-1}{\ln(x)}$

b)

Taylor $P_3(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3$

$f'(x) = x\left(1+x^2\right)^{-\frac{1}{2}}$ $\qquad f'(0) = 0$

$f''(x) = \left(1+x^2\right)^{-\frac{1}{2}} - x^2\left(1+x^2\right)^{-1\frac{1}{2}}$ $\qquad f''(0) = 1$

$f'''(x) = -3x\left(1+x^2\right)^{-1\frac{1}{2}} + 3x^3\left(1+x^2\right)^{-2\frac{1}{2}}$ $\qquad f'''(0) = 0$

Answer: $1 + \dfrac{1}{2}x^2$

4.

Differential equation is in standard form $\dfrac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$

Integrating factor $r = e^{\int P dx} = e^{\int x dx} = e^{\frac{1}{2}x^2}$

Multiply by v : $\quad e^{\frac{1}{2}x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x} + xe^{\frac{1}{2}x^2}y = xe^{\frac{1}{2}x^2}$

So $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(ye^{\frac{1}{2}x^2}\right) = xe^{\frac{1}{2}x^2}$

$$ye^{\frac{1}{2}x^2} = \int xe^{\frac{1}{2}x^2}dx = e^{\frac{1}{2}x^2} + c$$

$$y = 1 + ce^{-\frac{1}{2}x^2}$$

$y(0) = -6 \implies c = -7,$ answer $y = 1 - 7e^{-\frac{1}{2}x^2}$

5.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -0.18y \implies y(t) = c \cdot e^{-0.18t}$$

At time $t = 0$: $y(0) = c$

At what time t do we have $y(t) = \frac{1}{2}c$?

$$y(t) = \frac{1}{2}c \iff ce^{-0.18t} = \frac{1}{2}c \iff e^{-0.18t} = \frac{1}{2}$$

$$e^{-0.18t} = \frac{1}{2} \iff -0.18t = \ln\left(\frac{1}{2}\right) = -\ln(2) \iff t = \frac{\ln(2)}{0.18}\text{(days)}$$

6. a)

$z = 2\sqrt{3} - 2i$ In polar form: $z = 4e^{-\frac{1}{6}\pi i}$

$w = -1 + i$ In polar form: $w = \sqrt{2}e^{\frac{3}{4}\pi i}$

$$z \cdot w = 4\sqrt{2}e^{-\frac{1}{6}\pi i + \frac{3}{4}\pi i} = 4\sqrt{2}e^{\frac{7}{12}\pi i}$$

$$\frac{z}{w} = \frac{4}{\sqrt{2}}e^{-\frac{1}{6}\pi i - \frac{3}{4}\pi i} = 2\sqrt{2}e^{-\frac{11}{12}\pi i}$$

or

$$z \cdot w = 2 - 2\sqrt{3} + i\left(2 + 2\sqrt{3}\right)$$

$$\frac{z}{w} = -1 - \sqrt{3} + i\left(1 - \sqrt{3}\right)$$

b)

With $z = re^{i\Theta}$ we have

$$z^3 + 1 = 0 \iff z^3 = -1 \iff r^3 e^{i3\Theta} = e^{i\pi}$$

So $r = 1$ and $3\Theta = \pi + k \cdot 2\pi$

Solutions: $z_1 = e^{i\frac{1}{3}\pi}$

$$z_2 = e^{i\left(\frac{1}{3}\pi + \frac{2}{3}\pi\right)} = -1$$

$$z_3 = e^{i\left(\frac{1}{3}\pi + \frac{4}{3}\pi\right)} = e^{i\frac{5}{3}\pi}$$

7.

Solve the homogeneous equation $y'' + 2y' + 5y = 0$

Characteristic equation: $r^2 + 2r + 5 = 0 \iff r_{1,2} = -1 \pm 2i$

$y_H(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$

(Try) particular solution $y_P(x) = \text{constant } c \implies 5c = 20 \implies c = 4$

General solution: $y(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x) + 4 \quad (c_1, c_2 \in \mathbb{R})$

$y(0) = c_1 \cdot 1 \cdot 1 + c_2 \cdot 1 \cdot 0 + 4 = 4 \implies c_1 = 0$

$y'(x) = \big(c_2 e^{-x} \sin(2x) + 4\big)' = -c_2 e^{-x} \sin(2x) + 2c_2 e^{-x} \cos(2x)$

$y'(0) = -c_2 \cdot 1 \cdot 0 + 2 \cdot c_2 \cdot 1 \cdot 1 = 4 \implies c_2 = 2$

Answer: $y(x) = 2e^{-x} \sin(2x) + 4$