Answer sheet Calculus 1B

Friday January 8th, 2021, 13:45 – 15:45

Do not remove the staple!

Session [01-B]

Name : — Answers —

Programme : ____________________

Student number : ____________________

1. [3 pt] Select the correct answer:

right hand end point

$$S_n = \sum_{k=1}^{n} f(x_k)\cdot\Delta x \overset{\downarrow}{=} \sum_{k=1}^{n} f(k\cdot\Delta x)\cdot\Delta x =$$

$$\overset{\Delta x = \frac{3}{n}}{=} \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\cdot\frac{3}{n} = \sum_{k=1}^{n}\left(\frac{3k}{n}\right)^2 e^{-\frac{3k}{n}}\cdot\frac{3}{n} =$$

$$= \sum_{k=1}^{n} \frac{27k^2 e^{-\frac{3k}{n}}}{n^3}$$

- [x] A) $\displaystyle\sum_{k=1}^{n} \frac{27k^2 e^{-\frac{3k}{n}}}{n^3}$
- [ ] B) $\displaystyle\sum_{k=1}^{n} \frac{8k^2 e^{-\frac{2k}{n}}}{n^3}$
- [ ] C) $\displaystyle\sum_{k=1}^{n} \frac{(k-1)^2 e^{-\frac{3k-1}{n}}}{n^3}$
- [ ] D) $\displaystyle\sum_{k=1}^{n} \frac{8(k-1)^2 e^{-\frac{2(k-1)}{n}}}{n^3}$
- [ ] E) $\displaystyle\sum_{k=1}^{n} \frac{8k^2 e^{-\frac{3k}{n}}}{n^3}$
- [ ] F) $\displaystyle\sum_{k=1}^{n} \frac{k^2 e^{-\frac{3k}{n}}}{n^3}$
- [ ] G) $\displaystyle\sum_{k=1}^{n} \frac{27(k-1)^2 e^{-\frac{3(k-1)}{n}}}{n^3}$
- [ ] H) $\displaystyle\sum_{k=1}^{n} \frac{8\left(k-\frac{1}{2}\right)^2 e^{-\frac{2\left(k-\frac{1}{2}\right)}{n}}}{n^3}$

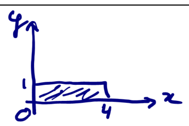
$y=\sqrt{16-x^2} \Rightarrow$
$y^2 = 16 - x^2 \Leftrightarrow$
$x^2+y^2=16$, circle
$M(0,0)$, $r=4$.

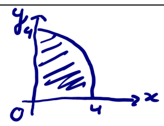
2. [2 pt] Write only your final answer in the frame below.

$$\int_0^4 1\cdot dx + \int_0^4 \sqrt{16-x^2}\,dx =$$

The integral equals:

$4\pi + 4$





$$= 1\cdot 4 + \frac{1}{4}\cdot\pi\, 4^2$$

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3. [2 pt] Give a full calculation/argumentation in the frame below.

$$\frac{d}{dx}\int_0^{2+\cos x} e^{t^2}\,dt =$$

$$= \frac{d}{dx}\Big[F(t)\Big]_0^{2+\cos x} \quad \text{with } F'(t) = e^{t^2} =: f(t)$$

$$= \frac{d}{dx}\left\{F(2+\cos x) - F(0)\right\} =$$

$$= \frac{d}{d(2+\cos x)}\left\{F(2+\cos x)\right\} \cdot \frac{d(2+\cos x)}{dx} =$$

$$= f(2+\cos x)\cdot -\sin x = -\sin x \cdot e^{(2+\cos x)^2}$$

$$= -\sin x \cdot e^{(2+\cos x)^2}$$

OR

Assume $f(x) := \int_0^{2+\cos x} e^{t^2}\,dt$, then $f(x) = g(2+\cos x)$ where

$$g(u) = \int_0^u e^{t^2}\,dt$$

Now $g'(u) = e^{u^2}$, so

$$f'(x) = g'(2+\cos x)\cdot -\sin x = -\sin x \cdot e^{(2+\cos x)^2}$$

4. [2 pt] Write only your final answer in the frame below.

The integral equals:

$$\frac{1}{3}\sin\left(1+x^3\right) + C$$

$$\int x^2\cos(1+x^3)\,dx = \int \cos(1+x^3)\,d\left(\tfrac{1}{3}(1+x^3)\right) = \int \tfrac{1}{3}\cos(1+x^3)\,d(1+x^3) =$$

$$= \left[\tfrac{1}{3}\sin(1+x^3)\right] = \tfrac{1}{3}\sin(1+x^3) + C$$

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5. [5 pt] Give a full calculation/argumentation in the frame below.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{p\to\infty} \int_1^p \frac{\ln x}{x^2}\,dx$$

For $p>1$:

$$\int_1^p \frac{\ln x}{x^2}\,dx =$$

$$= \left[-\frac{\ln x}{x}\right]_1^p - \int_1^p -x^{-2}\,dx =$$

$$= \left[-\frac{\ln x}{x}\right]_1^p - \left[\frac{1}{x}\right]_1^p =$$

$$= -\frac{\ln p}{p} + \frac{\ln(1)}{1} - \frac{1}{p} + 1 .$$

And finally

$$\lim_{p\to\infty} -\frac{\ln p}{p} - \frac{1}{p} + 1 = 1$$

Continue on the next page.

6. [3 pt] Select the correct answer:

- ○ A) $\frac{21}{5}$
- ○ B) $\frac{63}{15}$
- ○ C) $\frac{7}{15}$
- ○ D) $\frac{20}{15}$
- ○ E) $\frac{4}{3}$
- ○ F) $\frac{4}{5}$
- ○ G) $\frac{7}{3}$
- ● H) $\frac{83}{15}$

$$\sum_{n=0}^{\infty} 3\cdot\frac{2^n}{7^n} + \frac{1}{4^n} = 3\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n =$$

$$= 3\cdot\frac{1}{1-\frac{2}{7}} + \frac{1}{1-\frac{1}{4}} = 3\cdot\frac{1}{\frac{5}{7}} + \frac{1}{\frac{3}{4}} = \frac{21}{5} + \frac{4}{3} = \frac{83}{15}$$

↑ Geometric series $|x|<1$

7. [4 pt] Give a full calculation/argumentation in the frame below.

$$T_4(x) = f(1) + \frac{f^{(1)}(1)}{1!}(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

	Function	@ $x=1$
f	$\ln x$	0
$f^{(1)}$	$\frac{1}{x}$	1
$f^{(2)}$	$-\frac{1}{x^2}$	-1
$f^{(3)}$	$\frac{2}{x^3}$	2
$f^{(4)}$	$-\frac{6}{x^4}$	-6

So

$$T_4(x) = x-1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

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8
4p

$$\begin{cases} \frac{dy}{dx} = \frac{\sin x}{x^2} - \frac{2y}{x}, \quad x>0 \\ y(\frac{\pi}{2}) = 0 \end{cases}$$

Rewriting: $\frac{dy}{dx} + \frac{2}{x} \cdot y = \frac{\sin x}{x^2} \Rightarrow$

$P(x) = \frac{2}{x}$ and $Q(x) = \frac{\sin x}{x^2}$

- $M(x) = e^{\int P(x)dx}$

$$\int P(x)dx = \int \frac{2}{x}dx = 2\ln|x| \overset{x>0}{=} 2\ln(x)$$

$$M(x) = e^{2\ln(x)} = (e^{\ln(x)})^2 = x^2$$

- $y(x) = \frac{1}{M(x)} \int Q(x)M(x)\,dx + \frac{c}{M(x)} =$

$$= \frac{1}{x^2}\int \frac{\sin x}{\cancel{x^2}} \cdot \cancel{x^2}\,dx + \frac{c}{x^2} =$$

$$= \frac{1}{x^2}\left[-\cos x\right] + \frac{c}{x^2} = \frac{c}{x^2} - \frac{\cos x}{x^2}$$

- $y(\frac{\pi}{2}) = 0 \Leftrightarrow \frac{c}{(\frac{\pi}{2})^2} - \frac{\cos(\frac{\pi}{2})}{(\frac{\pi}{2})^2} = 0 \Leftrightarrow c - \underbrace{\cos(\frac{\pi}{2})}_{0} = 0$

$\Leftrightarrow c - 0 = 0 \Leftrightarrow c = 0$

Therefore $y(x) = -\frac{\cos x}{x^2}$.

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8. [4 pt] Write only your final answer in the frame below.

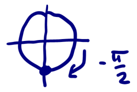
The solution is:

$$y(x) = -\frac{\cos x}{x^2}$$

9. [2 pt] Select the correct answer:

- ○ A) $-2\,e^{-i\frac{\pi}{6}}$
- ○ B) $2\,e^{i\frac{\pi}{6}}$
- ○ C) $-\sqrt{3}+i$
- ○ D) $-\sqrt{3}-i$
- ● E) $2\,e^{-i\frac{\pi}{6}}$
- ○ F) $\sqrt{3}+i$
- ○ G) $2\,e^{-i\frac{5\pi}{6}}$
- ○ H) $-2\,e^{i\frac{\pi}{6}}$

$z = -i = e^{-i\frac{\pi}{2}}$

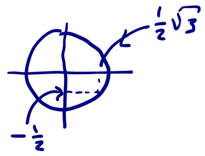


$\omega = 2 \cdot e^{i\frac{\pi}{3}}$

$z \cdot \omega = e^{-i\frac{\pi}{2}} \cdot 2\,e^{i\frac{\pi}{3}} =$

$= 2\,e^{i\left(\frac{\pi}{3}-\frac{\pi}{2}\right)} = 2\,e^{-i\frac{\pi}{6}}$

$\left(= 2\left(-\frac{1}{2}i + \frac{1}{2}\sqrt{3}\right) = \sqrt{3} - i\right)$



10. [3 pt] Give a full calculation/argumentation in the frame below.

$z^5 = -32i \iff$

$z^5 = 32\,e^{-i\frac{\pi}{2}}$

Then $z = r\,e^{i\varphi}$ with

$r = 2$

$\varphi = -\frac{\pi}{10} + \frac{4}{10}k\pi$ with $k = 0,1,2,3,4$

so

$z_0 = 2\,e^{-i\frac{\pi}{10}}$, $z_1 = 2\,e^{i\frac{3}{10}\pi}$, $z_2 = 2\,e^{i\frac{7}{10}\pi}$, $z_3 = 2\,e^{i\frac{11}{10}\pi}$, $z_4 = 2\,e^{i\frac{15}{10}\pi}$

$\left(z_3 = 2\,e^{-i\frac{9}{10}\pi},\ z_4 = 2\,e^{-i\frac{\pi}{2}}\right)$

Continue on the next page.

11. [6 pt] Give a full calculation/argumentation in the frame below.

$y'' - y = e^{-t}$ $\qquad y(0) = 0 \qquad y'(0) = 1$

Let $y = y_h + y_p$.

y_h: $(r^2-1)e^{rt} = 0 \iff r^2 = 1 \iff r = 1 \vee r = -1$, so

$y_h = C_1 e^{t} + C_2 e^{-t}$.

y_p: Let $y_p = \underset{!}{At}\, e^{-t}$, then

$y_p' = Ae^{-t} - Ate^{-t}$

$y_p'' = -Ae^{-t} - (Ae^{-t} - Ate^{-t}) =$

$= -2Ae^{-t} + Ate^{-t}$, and so

$y_p'' - y_p = -2Ae^{-t} + \cancel{Ate^{-t}} - \cancel{Ate^{-t}} = -2Ae^{-t}$

Now $-2Ae^{-t} = e^{-t} \quad \forall t \Rightarrow A = -\frac{1}{2}$ and thus

$y_p = -\frac{1}{2} t e^{-t}$

Finally: $y = C_1 e^{t} + C_2 e^{-t} - \frac{1}{2} t e^{-t}$

Initial values: $y' = C_1 e^{t} - C_2 e^{-t} - \frac{1}{2} e^{-t} + \frac{1}{2} t e^{-t}$

$y(0) = 0 \Rightarrow \quad C_1 + C_2 = 0$

$y'(0) = 1 \Rightarrow \quad C_1 - C_2 = 1\frac{1}{2} \quad +$

$2C_1 = \frac{3}{2} \Rightarrow C_1 = \frac{3}{4}$,

$C_2 = -\frac{3}{4}$

And so

$y = \frac{3}{4} e^{t} - \frac{3}{4} e^{-t} - \frac{1}{2} t e^{-t}$

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