Answer sheet Calculus 1B

Friday January 8th, 2021, 09:00 – 11:00

Do not remove the staple!

Session [01-A]

Name : — Answers —

Programme :

Student number :

1. [3 pt] Select the correct answer:

Left hand end point

$$S_n = \sum_{k=1}^{n} f(x_k)\cdot\Delta x = \sum_{k=1}^{n} f((k-1)\Delta x)\cdot\Delta x =$$

$$= \sum_{k=1}^{n} f\left((k-1)\cdot\frac{2}{n}\right)\cdot\frac{2}{n} = \sum_{k=1}^{n} \frac{4\cdot\left(\frac{k-1}{n}\cdot 2\right)}{\frac{(k-1)^2\cdot 4}{n^2}+1}\cdot\frac{2}{n} =$$

$\Delta x = \frac{2}{n}$

$$= \dots = \sum_{k=1}^{n} \frac{16(k-1)}{4(k-1)^2+n^2}$$

- ○ A) $\displaystyle\sum_{k=1}^{n} \frac{16(k-1)}{2(k-1)^2+n^2}$
- ○ B) $\displaystyle\sum_{k=1}^{n} \frac{16k-8}{(1-2k)^2+n^2}$
- ○ C) $\displaystyle\sum_{k=1}^{n} \frac{8k}{4k^2+n^2}$
- ○ D) $\displaystyle\sum_{k=1}^{n} \frac{8k-4}{(1-2k)^2+n^2}$
- ● E) $\displaystyle\sum_{k=1}^{n} \frac{16(k-1)}{4(k-1)^2+n^2}$
- ○ F) $\displaystyle\sum_{k=1}^{n} \frac{16k}{4k^2+n^2}$
- ○ G) $\displaystyle\sum_{k=1}^{n} \frac{8(k-1)}{4(k-1)^2+n^2}$
- ○ H) $\displaystyle\sum_{k=1}^{n} \frac{16k-8}{4(k-1)^2+n^2}$

$y=\sqrt{7-(x-4)^2} \Rightarrow$

$y^2 = 7-(x-4)^2 \Leftrightarrow$

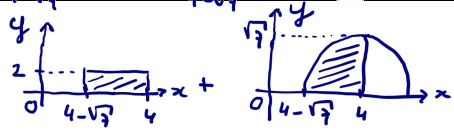
$(x-4)^2+y^2=7$, so circle with $M(4,0)$ and $r=\sqrt{7}$.

2. [2 pt] Write only your final answer in the frame below.

$$\int_{4-\sqrt{7}}^{4} 2\,dx + \int_{4-\sqrt{7}}^{4} \sqrt{7-(x-4)^2}\,dx = \bullet$$

The integral equals:

$\frac{7}{4}\pi + 2\sqrt{7}$



$$\bullet = \sqrt{7}\cdot 2 + \frac{1}{4}\cdot\pi(\sqrt{7})^2$$

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3. [2 pt] Give a full calculation/argumentation in the frame below.

$f(t) := \frac{1}{t^4+1}$, $F'(t) = f(t)$.

$$\frac{d}{dx}\int_{x^2}^{0} f(t)\,dt = \frac{d}{dx}\Big[F(t)\Big]_{x^2}^{0} = \frac{d}{dx}\Big(F(0) - F(x^2)\Big) =$$

$$= \frac{d}{dx}\left\{-F(x^2)\right\} = -\frac{d}{d(x^2)}F(x^2)\cdot\frac{d(x^2)}{dx} =$$

$$= -f(x^2)\cdot 2x = -\frac{2x}{x^8+1}.$$

OR

Assume $f(x) := \int_{x^2}^{0}\frac{1}{t^4+1}\,dt$, then $f(x) = g(x^2)$ where

$$g(u) = \int_u^0 \frac{1}{t^4+1}\,dt = -\int_0^u \frac{1}{t^4+1}\,dt.$$

Now $g'(u) = -\frac{1}{u^4+1}$, so

$$f'(x) = g'(x^2)\cdot 2x = -\frac{1}{x^8+1}\cdot 2x.$$

4. [4 pt] Write only your final answer in the frame below.

$$\int_0^{\sqrt{3}}\frac{\arctan^3(x)}{1+x^2}\,dx = \int_0^{\sqrt{3}}\arctan^3(x)\,d(\arctan(x)) =$$

The integral equals:

$\frac{\pi^4}{324}$

$$= \left[\frac{1}{4}\arctan^4(x)\right]_0^{\sqrt{3}} = \frac{1}{4}\left(\arctan^4(\sqrt{3}) - \arctan^4(0)\right) =$$

$$= \frac{1}{4}\left(\left(\frac{\pi}{3}\right)^4 - 0^4\right) = \frac{\pi^4}{324}$$

Continue on the next page.

5. [4 pt] Give a full calculation/argumentation in the frame below.

$$I := \int e^{2x} \sin x \, dx = \int \sin x \, d\left(\tfrac{1}{2}e^{2x}\right) =$$

$$= \left[\sin x \cdot \tfrac{1}{2}e^{2x}\right] - \int \tfrac{1}{2} e^{2x} \, d(\sin x) =$$

$$= \left[\sin x \cdot \tfrac{1}{2}e^{2x}\right] - \tfrac{1}{2}\int e^{2x} \cos x \, dx =$$

$$= [\cdot\cdot] - \tfrac{1}{2}\int \cos x \, d\left(\tfrac{1}{2}e^{2x}\right) =$$

$$= [\cdot\cdot] - \tfrac{1}{2}\left\{\left[\cos x \cdot \tfrac{1}{2}e^{2x}\right] - \int \tfrac{1}{2}e^{2x} \, d(\cos x)\right\} =$$

$$= \left[\sin x \cdot \tfrac{1}{2}e^{2x} - \tfrac{1}{2}\cos x \cdot \tfrac{1}{2}e^{2x}\right] + \tfrac{1}{2}\int \tfrac{1}{2}e^{2x} \cdot -\sin x \, dx =$$

$$= \left[\tfrac{1}{2}e^{2x}\left(\sin x - \tfrac{1}{2}\cos x\right)\right] - \tfrac{1}{4}\int e^{2x} \sin x \, dx$$

So $I = [\cdots] - \tfrac{1}{4} I \Leftrightarrow \tfrac{5}{4} I = [\cdots] \Leftrightarrow I = \tfrac{4}{5}[\cdots]$

Finally $I = \tfrac{2}{5} e^{2x}\left(\sin x - \tfrac{1}{2}\cos x\right) + C$

Continue on the next page.

6. [2 pt] Give a full calculation/argumentation in the frame below.

$$\sum_{n=0}^{\infty} 8 \cdot \frac{3^n}{4^{2+n}} = \sum_{n=0}^{\infty} 8 \cdot \frac{3^n}{4^2 \cdot 4^n} = \sum_{n=0}^{\infty} \frac{8}{16} \cdot \frac{3^n}{4^n} =$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \underset{\text{Geometric Series}}{=}$$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{3}{4}} =$$

$$= 2$$

7. [4 pt] Give a full calculation/argumentation in the frame below.

See next page.

Continue on the next page.

$$T_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 =$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3 \quad (*)$$

$f(x) = e^x \cdot x^2 \qquad =: u \cdot v \qquad u = e^x \qquad v = x^2$

$u' = e^x \qquad v' = 2x$

$f'(x) = u'v + uv' =$

$= e^x \cdot x^2 + e^x \cdot 2x =$

$= e^x(x^2 + 2x) \qquad =: u \cdot v \qquad u = e^x \qquad v = x^2 + 2x$

$u' = e^x \qquad v' = 2x + 2$

$f''(x) = u'v + uv' =$

$= e^x(x^2+2x) + e^x(2x+2) =$

$= e^x(x^2 + 4x + 2) \qquad =: uv \qquad u = e^x \qquad v = x^2 + 4x + 2$

$u' = e^x \qquad v' = 2x + 4$

$f'''(x) = u'v + uv' =$

$= e^x(x^2+4x+2) + e^x(2x+4)$

$= e^x(x^2 + 6x + 6)$

	Function	@ $x=1$
f	$e^x \cdot x^2$	e
f'	$e^x(x^2+2x)$	$3e$
f''	$e^x(x^2+4x+2)$	$7e$
f'''	$e^x(x^2+6x+6)$	$13e$,

So, with (*):

$$T_3(x) = e + 3e(x-1) + \frac{7e}{2}(x-1)^2 + \frac{13e}{6}(x-1)^3$$

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$\boxed{8}$

4pt

$$\begin{cases} \frac{dy}{dx} = \frac{y}{x+1} + (x+1)^2, \quad x > -1 \\ y(0) = 1 \end{cases}$$

$\frac{dy}{dx} + \left(-\frac{1}{x+1}\right) y = (x+1)^2$, so $P(x) = -\frac{1}{x+1}$ and $Q(x) = (x+1)^2$

- $M(x) = e^{\int P(x)\,dx}$

$\int P(x)\,dx = -\ln|x+1| \overset{x>-1}{=} -\ln(x+1)$, so

$M(x) = e^{-\ln(x+1)} = \left(e^{\ln(x+1)}\right)^{-1} = (x+1)^{-1} = \frac{1}{x+1}$

- $y(x) = \frac{1}{M(x)} \int Q(x) M(x)\,dx + \frac{C}{M(x)} =$

$= (x+1) \int (x+1)^2 \cdot \frac{1}{x+1}\,dx + C(x+1) =$

$= (x+1) \int (x+1)\,dx + C(x+1) = \frac{1}{2}(x+1)^3 + C(x+1)$

- $y(0) = 1 \iff \frac{1}{2} \cdot 1^3 + C \cdot 1 = 1 \iff C = \frac{1}{2}$, so

$y(x) = \frac{1}{2}(x+1)^3 + \frac{1}{2}(x+1) = \ldots$ see next page.

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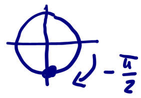
8. [4 pt] Write only your final answer in the frame below.

The solution is:

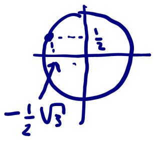
$y(x) = (x+1)\left(\frac{1}{2}x^2 + x + 1\right)$ OR $y(x) = \frac{1}{2}\left(x^3 + 3x^2 + 4x + 2\right)$

9. [3 pt] Select the correct answer:

$z = 2e^{i\cdot\frac{\pi}{3}}$ $\quad \omega = -i = e^{-i\frac{\pi}{2}}$



$\frac{z}{\omega} = \frac{2e^{i\frac{\pi}{3}}}{e^{-i\frac{\pi}{2}}} = 2e^{i\left(\frac{\pi}{3}+\frac{\pi}{2}\right)} = 2^{i\frac{5}{6}\pi}$



$= 2\left(-\frac{1}{2}\sqrt{3} + \frac{1}{2}i\right) =$

$= -\sqrt{3} + i$

- ○ A) $\sqrt{3} + i$
- ○ B) $-\sqrt{3} + 2i$
- ○ C) $\sqrt{3} - i$
- ○ D) $-2\sqrt{3} + i$
- ○ E) $-\sqrt{3} - i$
- ○ F) $\sqrt{3} + 2i$
- ○ G) $\frac{1}{2}\sqrt{3} + i$
- ● H) $-\sqrt{3} + i$

10. [2 pt] Give a full calculation/argumentation in the frame below.

$z^2 + 6z + 12 = 0$ $\qquad D = 36 - 4.12 = -12 = 12i^2$

$z_{1,2} = \frac{-6 \pm i\sqrt{12}}{2} =$

$= -3 \pm \frac{1}{2}i\sqrt{12}$

OR

$z_{1,2} = \frac{-6 \pm 2i\sqrt{3}}{2} =$

$= -3 \pm i\sqrt{3}$

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11. [6 pt] Give a full calculation/argumentation in the frame below.

$y'' + 5y' + 4y = t \qquad y(0) = 0 \qquad y'(0) = 0$

Let $y = y_h + y_p$

y_h: $(r^2 + 5r + 4)e^{rt} = 0 \iff r^2 + 5r + 4 = 0$

$\iff (r+1)(r+4) = 0 \iff r = -1 \vee r = -4$, so

$y_h = C_1 e^{-t} + C_2 e^{-4t}$

y_p: Let $y_p = At + B \Rightarrow y_p' = A$ and $y_p'' = 0$.

Then

$0 + 5A + 4(At + B) = t \quad \forall t \Rightarrow$

$4A = 1 \wedge 5A + 4B = 0 \iff$

$A = \frac{1}{4} \wedge B = -\frac{5}{16}$, so

$y_p = \frac{1}{4}t - \frac{5}{16}$

And thus: $y = C_1 e^{-t} + C_2 e^{-4t} + \frac{1}{4}t - \frac{5}{16}$

Initial values:

- $y' = -C_1 e^{-t} - 4C_2 e^{-4t} + \frac{1}{4}$

  $y'(0) = -C_1 - 4C_2 + \frac{1}{4}$
- $y = C_1 e^{-t} + C_2 e^{-4t} + \frac{1}{4}t - \frac{5}{16}$

  $y(0) = C_1 + C_2 - \frac{5}{16}$

$$\begin{cases} -C_1 - 4C_2 = -\frac{1}{4} \\ C_1 + C_2 = \frac{5}{16} \; + \end{cases}$$

$-3C_2 = \frac{1}{16} \Rightarrow C_2 = -\frac{1}{48}, \; C_1 = \frac{1}{3}$

Finally: $y = \frac{1}{3}e^{-t} - \frac{1}{48}e^{-4t} + \frac{1}{4}t - \frac{5}{16}$

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