Exam (January 21, 2026, 13:45–16:45)

Every answer requires an explanation unless explicitly stated otherwise. The maximum score is 36 points; the grade is calculated by $1 + \frac{\#\text{points}}{4}$. To remind you, here is a list of notation used in this exam:

$\delta(G)$ minimal degree
$\kappa(G)$ (vertex) connectivity
$\lambda(G)$ edge connectivity
$\chi(G)$ colourability
$\mathrm{cl}(G)$ closure of G

Problem 1. Recall that a clique of a graph $G = (V, E)$ is a subset V' of V such that $vw \in E$ for all $v, w \in V'$. Let $\gamma(G)$ be the size of the largest clique in G.

(a) (3pt) Show that $\chi(G) \geq \gamma(G)$.
(b) (3pt) Is the inequality in (a) always an equality? Give a proof or a counterexample.

Problem 2. Let G be a graph. For each statement, give either a proof or a counterexample. For (b) and (c), you either need a proof that it holds for all $k \geq 1$, or a counterexample for a single k.

(a) (2pt) If G is bipartite, then $\mathrm{cl}(G)$ is bipartite.
(b) (2pt) If G is k-connected, then $\mathrm{cl}(G)$ is k-connected.
(c) (2pt) If G has a k-independent set, then $\mathrm{cl}(G)$ has a k-independent set.

Problem 3. Let G be a graph. A k-edge colouring is a map $E \to \{1, \ldots, k\}$ such that two edges get a different colour if they share a vertex. Consider the following decision problem:

EDGE-COL
Instance: graph G, integer $k \geq 2$
Question: Does G have a k-edge colouring?

(a) (4pt) Show that EDGE-COL $\leq_p$ COL.
(b) (2pt) Does this prove that EDGE-COL is in **NPC**? Why (not)?

The exam continues on the next page!

Problem 4. Let G be a connected graph.

(a) (2pt) Show that $n(\kappa(G) + \lambda(G)) \leq 4m$ (recall that $n = |V|$ and $m = |E|$).
(b) (2pt) Show that if equality holds in (a), then G is regular (i.e., all vertices have the same degree).
(b) (2pt) Give an example of a connected regular graph in which equality does not hold in (a).

Problem 5. (6pt) Let $k \geq 2$ be an integer. We say that a graph G has a k-star if there exist vertices $v, w_1, \ldots, w_k$ such that each edge vw_i is in E. We say that G has an induced k-star if it has a k-star such that the only edges between $v, w_1, \ldots, w_k$ are the vw_i. Consider the following two decision problems:

k-STAR
Instance: graph G, integer $k \geq 2$
Question: Does G have a k-star?

INDUCED k-STAR
Instance: graph G, integer $k \geq 2$
Question: Does G have an induced k-star?

Show that one of these problems is in **P**, and the other in **NPC**.

Problem 6. (6pt) Let G be a connected planar graph with ≥ 3 vertices and an Euler cycle. What are the possible values of $\delta(G)$? For each $k \geq 0$, either give an example of a connected, Eulerian, planar graph G with ≥ 2 vertices and $\delta(G) = k$, or show that such a G does not exist.