Linear Algebra Date : 4 June 2021 Time : 13.45 – 15.45

First read these instructions carefully:

This test contains 9 exercises. The complete solutions of Exercises 3, 5, 6, 7 and 9 must be accurately written down on a separate sheet (that is, not on the supplied answer sheet) including calculations and argumentation. For the other exercises you are only required to fill in the final answers on the answer sheet at the end of this test. You must hand in this answer sheet as well as your hand written solutions to Exercises 3, 5, 6, 7 and 9.

The use of electronic devices is not allowed.

1. *Fill in your final answer to this exercise on the supplied answer sheet.*
   Three equations for planes are given below. Determine the values of α which correspond to the planes intersecting in
   a) a point
   b) a line

   You do not need to find the point or the line.

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + \alpha z &= 3 \\ x + \alpha y + 3z &= 2 \end{aligned}$$

2. *Fill in your final answer to this exercise on the supplied answer sheet.*
   You are given matrix $A \in \mathbb{R}^{4\times 5}$:

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

   a) What is the dimension of Null A?
   b) What is the dimension of Col A?

3. *Use a separate sheet and include clear argumentation and calculation.*
   Let V be the vector space consisting of all 2×2 matrices. The set S is defined as

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V \;\middle|\; a + d = 0 \right\}$$

   Is S a linear subspace of V? Motivate your answer.

4. *Fill in your final answer to this exercise on the supplied answer sheet.*
   Let A and B be two 2×2 matrices such that $\det A = 2$ and $\det B = -3$. Vector $\mathbf{v} \in \mathbb{R}^3$ is described as

$$\mathbf{v} = \begin{pmatrix} \det(AB) \\ \det(2B) \\ \det(A^{-1}) \end{pmatrix}.$$

   Evaluate the components of $\mathbf{v}$.

5. *Use a separate sheet and include clear argumentation and calculation.*
Let C be a 2×2 matrix of real numbers. The trace of a matrix is defined as the sum of the entries on the main diagonal. For each of the two statements below say whether it is TRUE or FALSE. If true prove it. If false give a counterexample.
a) $\det(C^2)$ is non-negative. b) $\operatorname{trace}(C^2)$ is non-negative.

6. *Use a separate sheet and include clear argumentation and calculation.*
Matrix A has eigenvalues 0, 1 and 2 with associated eigenspaces

$$E_0 = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\},\quad E_1 = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\},\quad E_2 = \operatorname{Span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

a) Determine Null A.
b) Determine Col A.

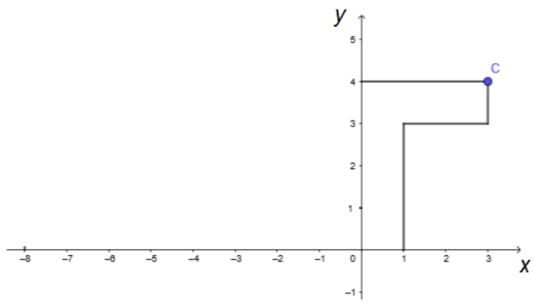
7. *Use a separate sheet and include clear argumentation and calculation.*
You are given matrix $M = \begin{pmatrix}2 & 0 & -1\\0 & 2 & 0\\1 & 0 & 2\end{pmatrix}$.
Find all the eigenvalues and the corresponding eigenspaces of M.

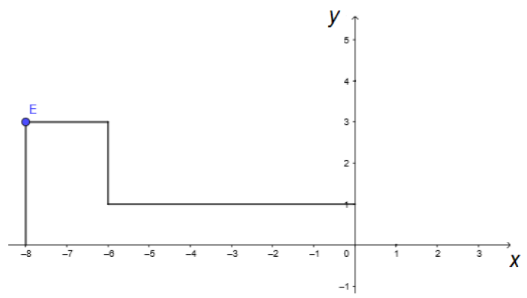
8. *Fill in your final answer to this exercise on the supplied answer sheet.*
The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\2x_1 + x_2\\3x_1 + x_2\end{pmatrix}.$$

Consider whether T is one-to-one and whether T is onto. Which of the following statements is correct?
A) T is both onto and one-to-one.
B) T is one-to-one, but not onto.
C) T is onto, but not one-to-one.
D) T is neither onto nor one-to-one.

9. *Use a separate sheet and include clear argumentation and calculation.*
Find a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ that transforms the shape including C (3,4) as shown on the figure above into the shape including E (−8,3) as shown on the figure below.





Points per question:

Question **1**: 2+2, Question **2**: 2+2, Question **3**:5, Question **4**: 3, Question **5**:4, Question **6**:4, Question **7**: 5, Question **8**: 3, Question **9**: 4

Final grade = (points+4)/4