Discrete Mathematics for Computer Science, 27 October 2017
Solutions/Correction standard

1. Basis step for $n = 0$, $n = 1$ and $n = 2$:
   $a_0 = 1 \leq \left[\frac{8}{3}\right]^0$, $\quad a_1 = 2 \leq \left[\frac{8}{3}\right]^1$ and $\quad a_2 = 7 \leq \frac{64}{9} \leq \left[\frac{8}{3}\right]^2$.

   Induction step:

   Let $k \geq 2$ and suppose that: $a_p \leq \left[\frac{8}{3}\right]^p$ for all $\quad 0 \leq p \leq k$ (Induction Hypothesis: IH).

   We must show that IH implies: $a_{k+1} \leq \left[\frac{8}{3}\right]^{k+1}$.

   Well, we have: $a_{k+1} = 2a_k + a_{k-1} + 2a_{k-2}$,

   and by IH this is less than or equal to $\quad 2\left[\frac{8}{3}\right]^k + \left[\frac{8}{3}\right]^{k-1} + 2\left[\frac{8}{3}\right]^{k-2}$.

   So it suffices to show that: $\quad 2\left[\frac{8}{3}\right]^k + \left[\frac{8}{3}\right]^{k-1} + 2\left[\frac{8}{3}\right]^{k-2} \leq \left[\frac{8}{3}\right]^{k+1}$.

   Dividing both sides by $\left[\frac{8}{3}\right]^{k-2}$ yields: $\quad 2\left[\frac{8}{3}\right]^2 + \left[\frac{8}{3}\right] + 2 \leq \left[\frac{8}{3}\right]^3$,

   which is obviously true since: $\quad 2 \cdot \frac{64}{9} + \frac{8}{3} + 2 = \frac{510}{27} \leq \frac{512}{27} = \left[\frac{8}{3}\right]^3$.

2. Suppose that f is onto and $g \circ f$ is one-to-one.
   Let $b_1, b_2 \in B$ be such that $g(b_1) = g(b_2)$. We must show that $b_1 = b_2$.
   Well, since f is onto, there exist $a_1, a_2 \in A$ with $f(a_1) = b_1$ and $f(a_2) = b_2$.
   Then we have $(g \circ f)(a_1) = g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2)) = (g \circ f)(a_2)$.
   So $(g \circ f)(a_1) = (g \circ f)(a_2)$. Now $g \circ f$ is one-to-one implies that $a_1 = a_2$.
   So $b_1 = f(a_1) = f(a_2) = b_2$. Hence g is one-to-one.

3. (i) R is reflexive since $(a,b)R(a,b)$ for all $(a,b) \in A$, because $a + b = b + a$.

   (ii) R is symmetric since for all $(a,b),(c,d) \in A$, $(a,b)R(c,d)$ implies $(c,d)R(a,b)$, because if $a + d = b + c$, then also $c + b = d + a$.

   (iii) R is transitive since for all $(a,b),(c,d),(e,f) \in A$, $(a,b)R(c,d)$ and $(c,d)R(e,f)$ implies $(a,b)R(e,f)$, because if both $a+d = b+c$ and $c+f = d+e$, then $a+c+d+f = b+c+d+e$. So $a + f = b + e$.

   (iv) The blocks of the partion of A induced by R are the equivalence classes of R.
   We have:
   $[(0,0)] = \{(0,0),(1,1),(2,2)\} = [(1,1)] = [(2,2)]$; $\quad [(0,1)] = \{(0,1),(1,2)\} = [(1,2)]$;
   $[(0,2)] = \{(0,2)\}$; $\quad [(1,0)] = \{(1,0),(2,1)\} = [(2,1)]$ and $\quad [(2,0)] = \{(2,0)\}$.
   So the partition is: $\quad \{[(0,0)],[(0,1)],[(0,2)],[(2,1)],[(2,0)]\}$
   $= \{\{(0,0),(1,1),(2,2)\},\{(0,1),(1,2)\},\{(0,2)\},\{(1,0),(2,1)\},\{(2,0)\}\}$.