Solutions to Calculus 1B exam 14.01.2022

Second capital letter in version tag indicates the version of the exam (A, B, C or D):

A: xwv4BcAz B: zcKokeBL C: O653iyCG D: enNDkZgT

① $f(x) = e^x$ Interval $[1,4]$ Left-hand endpoint

n equal subintervals : $\Delta x_k = \frac{3}{n}$

Riemann sum : k^{th} subinterval: $\left[1+(k-1)\cdot\frac{3}{n},\ 1+k\cdot\frac{3}{n}\right]$

$$= \left[\underbrace{1+\frac{3(k-1)}{n}}_{c_k},\ 1+\frac{3k}{n}\right]$$

$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

$$= \sum_{k=1}^{n} e^{1+\frac{3(k-1)}{n}}\cdot\frac{3}{n}$$

$$= \sum_{k=1}^{n} \frac{3}{n} e^{\frac{3(k-1)}{n}+1}$$

Version	A	B	C	D
Answer	E	D	E	D

② Let $y = \int_2^u \sin(t^2)\,dt$, with $u = \frac{1}{x}$.

Then $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}\int_2^u \sin(t^2)\,dt \cdot \frac{-1}{x^2}$$

$$= \sin(u^2)\cdot\frac{-1}{x^2} = -\frac{\sin\left(\frac{1}{x^2}\right)}{x^2}$$

③ $\displaystyle\int_1^2 \frac{1}{x\sqrt{x-1}}\,dx = \lim_{a\to 1^+} \int_a^2 \frac{1}{x\sqrt{x-1}}\,dx \underset{\uparrow}{=} \lim_{a\to 1^+} \int_{\sqrt{a-1}}^{1} \frac{1}{(u^2+1)\cancel{u}}\,2\cancel{u}\,du$

$u=\sqrt{x-1} \Rightarrow x = u^2+1$

$\frac{du}{dx} = \frac{1}{2\sqrt{x-1}} = \frac{1}{2u} \Rightarrow dx = 2u\,du$

$$= \lim_{a\to 1^+} \Big[2\arctan u\Big]_{u=\sqrt{a-1}}^{1}$$

$$= 2\arctan 1 - 2\arctan 0$$

$$= 2\cdot\frac{\pi}{4} - 2\cdot 0 = \frac{\pi}{2}.$$

④ $\displaystyle\int 4x^3 e^{x^2}\,dx \underset{\uparrow}{=} \int \underbrace{2u}_{f}\,\underbrace{e^u}_{g'}\,du$

$u = x^2$

$du = 2x\,dx \Rightarrow 4x^3\,dx = 2x^2\,du = 2u\,du$

$$= 2ue^u - \int 2e^u\,du$$

$$= 2ue^u - 2e^u + C$$

$$= 2e^u(u-1) + C$$

$$= 2e^{x^2}(x^2-1) + C$$

version A (See next page for answers for all versions)

⑤ a) $\sum_{n=1}^{\infty} 2\cdot\left(\frac{1-3x}{4}\right)^n$ Converges iff $\left|\frac{1-3x}{4}\right|<1$

$$\Leftrightarrow -1<\frac{1-3x}{4}<1 \Leftrightarrow -4<1-3x<4$$

$$\Leftrightarrow -5<-3x<3 \Leftrightarrow -1<x<\frac{5}{3}$$

So the interval of convergence is $-1<x<\frac{5}{3}$,

and the radius of convergence is $\frac{\frac{5}{3}-(-1)}{2}=\frac{4}{3}$.

b) Assuming $-1<x<\frac{5}{3}$, we find that

$$\sum_{n=1}^{\infty} 2\cdot\left(\frac{1-3x}{4}\right)^n = 2\cdot\underbrace{\sum_{n=1}^{\infty}\left(\frac{1-3x}{4}\right)^n}$$

first term $(n=1)$: $\frac{1-3x}{4}$

common ratio: $\frac{1-3x}{4}$

$$= 2\cdot\frac{\frac{1-3x}{4}}{1-\frac{1-3x}{4}}$$

$$= 2\cdot\frac{\frac{1-3x}{4}}{\frac{3+3x}{4}} = \frac{2-6x}{3+3x}.$$

first term $(n=0)$: 2

common ratio: $\frac{1-3x}{4}$

Alternatively: $$\sum_{n=1}^{\infty} 2\left(\frac{1-3x}{4}\right)^n = \underbrace{\sum_{n=0}^{\infty} 2\left(\frac{1-3x}{4}\right)^n} - 2$$

$$= \frac{2}{1-\frac{1-3x}{4}}-2 = \frac{2}{\frac{3+3x}{4}}-2 = \frac{8}{3+3x}-2$$

⑤ a) Answers all versions:

Version	Series	Interval	Radius	Sum
A	$\sum_{n=1}^{\infty} 2\cdot\left(\frac{1-3x}{4}\right)^n$	$-1 < x < \frac{5}{3}$	$\frac{4}{3}$	$\frac{2-6x}{3+3x}$
B	$\sum_{n=1}^{\infty} 3\cdot\left(\frac{1-2x}{4}\right)^n$	$-\frac{3}{2} < x < \frac{5}{2}$	2	$\frac{3-6x}{3+2x}$
C	$\sum_{n=1}^{\infty} 2\cdot\left(\frac{1-3x}{5}\right)^n$	$-\frac{4}{3} < x < 2$	$\frac{5}{3}$	$\frac{2-6x}{4+3x}$
D	$\sum_{n=1}^{\infty} 3\cdot\left(\frac{1-2x}{5}\right)^n$	$-2 < x < 3$	$\frac{5}{2}$	$\frac{3-6x}{4+2x}$

⑥ $f(x) = \sin^2 x$ $\qquad\qquad f\left(\frac{\pi}{2}\right) = 1$

Derivatives of f: $\qquad\qquad$ Evaluated at $x = \frac{\pi}{2}$:

$f'(x) = 2\sin x \cos x \quad = \sin(2x) \qquad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = 2\cos^2 x - 2\sin^2 x \quad = 2\cos(2x) \qquad f''\left(\frac{\pi}{2}\right) = -2$

$f'''(x) = -8\sin x \cos x \quad = -4\sin(2x) \qquad f'''\left(\frac{\pi}{2}\right) = 0$

$f^{(4)}(x) = -8\cos^2 x + 8\sin^2 x \quad = -8\cos(2x) \qquad f^{(4)}\left(\frac{\pi}{2}\right) = 8$

Taylor polynomial of order 4 generated by f at $x = \frac{\pi}{2}$:

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!}\left(x - \frac{\pi}{2}\right)^k$$

$$= 1 - \frac{2}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{8}{4!}\left(x - \frac{\pi}{2}\right)^4$$

$$= 1 - \left(x - \frac{\pi}{2}\right)^2 + \frac{1}{3}\left(x - \frac{\pi}{2}\right)^4$$

7 $x\dfrac{dy}{dx} = 2\sqrt{x} + 3y \quad , \quad y(1) = 0$

Standard form : $\dfrac{dy}{dx} - \dfrac{3}{x}y = \dfrac{2\sqrt{x}}{x}$

$$\frac{dy}{dx} \underbrace{- \frac{3}{x}}_{P(x)} y = \underbrace{2x^{-\frac{1}{2}}}_{Q(x)} \quad (*)$$

Integrating factor : $v(x) = e^{\int P(x)\,dx} = e^{\int -\frac{3}{x}\,dx}$

$= e^{-3\ln|x|} = e^{-3\ln(x)} = e^{\ln x^{-3}} = x^{-3} = \dfrac{1}{x^3}$

$x \geq 0$ due to $\sqrt{x}$ in DE

Multiplying (*) by $v(x)$ yields :

$$\frac{1}{x^3}\left(\frac{dy}{dx} - \frac{3}{x}y\right) = \frac{1}{x^3} \cdot 2x^{-\frac{1}{2}}$$

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = 2x^{-\frac{7}{2}}$$

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = 2x^{-\frac{7}{2}}$$

$$\frac{1}{x^3}y = \int 2x^{-\frac{7}{2}}\,dx$$

$$\frac{1}{x^3}y = -\frac{4}{5}x^{-\frac{5}{2}} + C$$

$$y = -\frac{4}{5}\sqrt{x} + Cx^3$$

Plugging in $y(1) = 0$:

$0 = -\frac{4}{5}\sqrt{1} + C \cdot 1^3$

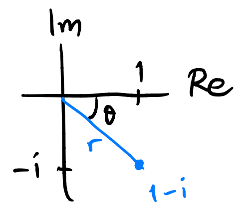
$\Rightarrow C = \frac{4}{5}$

So solution is

$$y = \frac{4}{5}\left(x^3 - \sqrt{x}\right)$$

version B (see next page for answers for all versions)
↓

(8) a) $z = 1 - i = re^{i\theta}$



$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = -\frac{\pi}{4}$

So $z = \sqrt{2}\, e^{-i\frac{\pi}{4}}$

b) The square roots of z are

$$\sqrt{\sqrt{2}}\, e^{i\left(-\frac{\pi}{4} + k\cdot 2\pi\right)\frac{1}{2}}$$

$$= \sqrt[4]{2}\, e^{i\left(-\frac{\pi}{8} + k\cdot\pi\right)}, \quad k \in \mathbb{Z}$$

Taking $k=0$ and $k=1$ yields the square roots

$$z_0 = \sqrt[4]{2}\, e^{-i\frac{\pi}{8}} \quad \text{and} \quad z_1 = \sqrt[4]{2}\, e^{\frac{7\pi}{8}}$$

c) $$W = \frac{1}{z^{10}} = \frac{1}{\left(\sqrt{2}\, e^{-i\cdot\frac{\pi}{4}}\right)^{10}} = \frac{1}{2^5 e^{-i\frac{5\pi}{2}}} = \frac{1}{32 e^{-i\frac{\pi}{2}}}$$

$$= \frac{1}{-32i} = \frac{i}{-32i^2} = \frac{1}{32} i$$

So $\mathrm{Re}(w) = 0$ and $\mathrm{Im}(w) = \frac{1}{32}$

⑧ Answers all versions:

Version	z	$re^{i\theta}$	Roots	$Re(w)$	$Im(w)$
A	$-1+i$	$\sqrt{2}e^{i\frac{3\pi}{4}}$	$\begin{cases} z_0 = \sqrt[4]{2}e^{i\frac{3\pi}{8}} \\ z_1 = \sqrt[4]{2}e^{i\frac{11\pi}{8}} \end{cases}$	0	$-\frac{1}{8}$
B	$1-i$	$\sqrt{2}e^{-i\frac{\pi}{4}}$	$\begin{cases} z_0 = \sqrt[4]{2}e^{-i\frac{\pi}{8}} \\ z_1 = \sqrt[4]{2}e^{i\frac{7\pi}{8}} \end{cases}$	0	$\frac{1}{32}$
C	$1-i$	$\sqrt{2}e^{-i\frac{\pi}{4}}$	$\begin{cases} z_0 = \sqrt[4]{2}e^{-i\frac{\pi}{8}} \\ z_1 = \sqrt[4]{2}e^{i\frac{7\pi}{8}} \end{cases}$	0	$-\frac{1}{8}$
D	$-1+i$	$\sqrt{2}e^{i\frac{3\pi}{4}}$	$\begin{cases} z_0 = \sqrt[4]{2}e^{i\frac{3\pi}{8}} \\ z_1 = \sqrt[4]{2}e^{i\frac{11\pi}{8}} \end{cases}$	0	$\frac{1}{32}$

⑨ $y'' - 6y' + 10y = 5 \quad , \quad y(0) = 0 \ , \ y'(0) = 1$

Characteristic equation: $\quad r^2 - 6r + 10 = 0$

Roots: $\quad r = \dfrac{6 \pm \sqrt{36-40}}{2} = \dfrac{6 \pm 2i}{2} = 3 \pm i$

So $y_h = e^{3x}\left(c_1 \cos x + c_2 \sin x\right)$

For the particular solution, we try

$$\left.\begin{aligned} y_p &= A \\ y_p' &= 0 \\ y_p'' &= 0 \end{aligned}\right\} \quad 5 = y_p'' - 6y_p' + 10y_p = 10A \Rightarrow A = \tfrac{1}{2}$$

General solution: $y(x) = e^{3x}\left(c_1 \cos x + c_2 \sin x\right) + \frac{1}{2}$

$\hookrightarrow y(0) = 0 \Rightarrow \quad 0 = c_1 + \frac{1}{2} \Rightarrow c_1 = -\frac{1}{2}$

$y'(x) = 3e^{3x}\left(c_1 \cos x + c_2 \sin x\right) + e^{3x}\left(-c_1 \sin x + c_2 \cos x\right)$

$\hookrightarrow y'(0) = 1 \Rightarrow \ 1 = 3c_1 + c_2 = -\frac{3}{2} + c_2 \Rightarrow c_2 = \frac{5}{2}$

So solution is $y(x) = e^{3x}\left(-\frac{1}{2}\cos x + \frac{5}{2}\sin x\right) + \frac{1}{2}$