Exam

Abstract Algebra

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April 12, 2024

1 Final answer / multiple choice questions

Question 1. Given are: $\sigma = (2356)$ (cyclic notation) and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 5 & 1 \end{pmatrix}$ (array notation). Find $\sigma^{-1}\tau$ and write your answer in array notation.

Question 2. How many abelian groups are there of order 100 up to isomorphism?

Question 3. Given is the homomorphism $GL_2(\mathbf{Z}) \to U(3)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to ad - bc \pmod 3$$

$GL_2(\mathbf{Z})$ is the multiplicative group of (2×2)-matrices with entries in $\mathbf{Z}$ and determinant 1 or -1. What is the index of the kernel in $GL_2(\mathbf{Z})$ of this homomorphism?

Question 4. What is the order of $((12)(345), 2)$ in $S_5 \times \mathbb{Z}_8$?

Question 5. Which elements in D_5 generate the subgroup $< r >$?

Question 6. What is the logarithm of (234) with base$= (123)(56)(134)$ in S_6?

Question 7. Given are the groups G, H and K. K is a subgroup of H and H is a subgroup of G ($K < H < G$). Which of these statements is always true? Give only one answer.

(a) The index of H in G is bigger than the index of K in G

(b) The index of K in H is bigger than the index of H in G

(c) The index of K in G is bigger than the index of K in H

(d) The index of K in H is bigger than the index of K in G

Question 8. Given is an abelian group G. H is a subgroup of G. What is true about H? Only give one answer.

(a) H is normal

(b) H is normal if H is cyclic, otherwise not normal

(c) H is normal if H has cyclic subgroups, otherwise not normal

(d) H is normal if H is the trivial subgroup, otherwise not normal

Question 9. In the group with presentation $< w, x, y, z | wxy, wx^{-1}, wy^{-1} >$, find a word of shortest length equivalent to $wwwzyxwx^{-1}$

Question 10. Which of the following groups is not isomorphic to $U(12)$? Only give one answer.

(a) $U(8)$

(b) $U(2) \times U(6)$

(c) $U(3) \times U(6)$

(d) $U(4) \times U(6)$

Question 11. Is $2x - 1$ in $\mathbf{Z}_4[x]$ a unit, zero divisor, both or neither? Only give one answer.

(a) $2x - 1$ is only a unit, not a zero divisor.

(b) $2x - 1$ is not a unit, only a zero divisor.

(c) $2x - 1$ is a unit and a zero divisor.

(d) $2x - 1$ is not a unit, nor a zero divisor.

Question 12. Given are $q(x) = x^3 + x^2 - x - 1$ and $p(x) = 2x^3 + x^2 - 4x + 3$. Are they irreducible in $\mathbf{Z}_5$? Only give one answer.

(a) Only $p(x)$ is irreducible.

(b) Only $q(x)$ is irreducible.

(c) Both $p(x)$ and $q(x)$ are irreducible.

(d) Both $p(x)$ and $q(x)$ are reducible.

Question 13. Given are $p(x) = x^4 - 2x^3 - x^2 - 2x + 1$ and $q(x) = x^3 - 2x^2 - 2x + 2$. What is the monic $gcd(p(x), q(x))$?

Question 14. Give the lowest integer which is a solution to the following system of congruences:

$$\begin{cases} x \equiv 2 \pmod 5 \\ x \equiv 5 \pmod{11} \end{cases}$$

Question 15. Given is the free group $F(S) = a, b, c$ and we define the homomorphism $\varphi : F(S) \to \mathbf{Z} \times S_5$ in which we have $\varphi(a) = (2, ())$, $\varphi(b) = (0, (345))$ and $\varphi(c) = (-2, (354))$. Solve: $\varphi(abccb^{-1}c^{-1}aa)$.

2 Questions with explanation

For these questions write down the full calculations.

Question 1. Given is the homomorphism $S_n \times \mathbf{R}^n \to \mathbf{R}^n$. The permutation given by S_n changes the order of $\mathbf{R}^n$ accordingly. This homomorphism is defined as follows:

$$(\sigma, (x_1, x_2, ...x_n)) \to (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, ..., x_{\sigma^{-1}(n)})$$

Example:

$$((123), (7, 10, 11)) \to (11, 7, 10)$$

(a) What are the orbits of $(1, 1, 1)$, $(1, 2, 2)$ and $(4, 5, 6)$?

(b) What is the stabilizer set of $(7, \pi, 7, \sqrt{2})$ and $(0, 0, 5, 0)$?

(c) Describe the fixed point set of this homomorphism. What is in it?

Question 2. Prove or disprove that $\mathbf{Z}[\sqrt{3}]/I$ is a field. It is given that $2+\sqrt{3} \in I$. Hint: compute $(2+\sqrt{3})(2-\sqrt{3})$.

Question 3. Let G and H be groups, and H is a subgroup of G. Given is: $[G : H] = 2$. Prove that for any $g \in G$, then $g^2 \in H$.