Course	:	Algebra	Date	:	Mod7
Module	:	2A	Time	:	2h 30min
Course code	:	202001362	Reference	:	ALGEBRA-2324-SAMPLETEST

Algebra

Sample Test – Mod7

Instructions

You will hand in the printed exam and answer form only. Any text outside of the answer form will not be considered.
Do not remove the staple.
If you run out of space, use the extra space at the end. Clearly refer to that space in the original box.

The use of electronic devices or personal notes is NOT allowed.

The exam consists of the following sections:

Final Answer / Multiple Choice (15 questions). *You shall write your answer in the box below or next to each question. Each correct answer is worth 1 point. Only a completely correct answer gains points. Each multiple choice questions has one correct answer among the given options.*

Questions with Explanation (3 questions). *Write a complete argumentation and explanation in the space provided at the end of the stapled exam booklet. Each question is worth 4 points.*

The total number of points is 27. The grade for p points is $1 + \frac{p}{3}$ rounded.

Notation / Terminology

- S_n – the symmetric group on $\{1, \ldots, n\}$
- $(\,)$ – the identity permutation in S_n
- $\mathrm{U}(n)$ – the multiplicative group of units of $\mathbb{Z}_n$
- $\mathrm{SL}_2(\mathbb{Z})$ – multiplicative group of (2×2)-matrices with entries in $\mathbb{Z}$ and determinant 1
- $\mathrm{SL}_2(\mathbb{Z}_2)$ – multiplicative group of (2×2)-matrices with entries in $\mathbb{Z}_2$ and determinant 1
- $\mathcal{F}(X)$ – the free group over the set X
- $\mathrm{Aut}(X)$ – the group of bijective functions on the set X.

DISCLAIMER

This sample test gives you an **example** of the kind of exercises you may find on your real test, and shows the approximate length of an exam, layout, instructions, point distribution, and question types (e.g. final answer, multiple choice, proofs, ...).

You should **not** expect the real tests to be a carbon copy of this sample test (or of each other). All the material covered during the course is examinable.

FA/MC	QwE1	QwE2	QwE3	Σ

Do not fill in this table.

Only for sorting.

Algebra

Sample Test – Mod7

Full name in BLOCK LETTERS ______________________

Student number ______________________

Programme ☐ AM ☐ TCS ☐ Other: ______________

FILL IN YOUR DATA AS SOON AS YOU RECEIVE THE EXAM

DO NOT REMOVE THE STAPLE

Final Answer / Multiple Choice **[15 points]**

1. What is the order of $(2, 4)$ in $\mathbb{Z}_{12} \times \mathbb{Z}_{30}$?

Answer:

2. What is the discrete logarithm of $(3\,5)(4\,6)$ with basis $(2\,1\,7)(3\,4\,5\,6)$ in S_7?

Answer:

3. In the group with presentation $\langle x, y, z \mid xx, yz, zz, yy\rangle$, find a word of shortest length equivalent to $zxyzxyyx$.

Answer:

4. List all the permutations that may generate the cyclic group $\langle(1\,8\,9\,2)\rangle \leq S_9$.

Answer:

5. Let $\mathcal{F}(T)$ be the free group over the set $T = \{x, y, z\}$, and let G be the subgroup of the Cartesian product $S_4 \times \mathbb{Z}_3$ generated by $\{((1\,2), 1),\ ((1\,3), 1),\ ((1\,2\,3\,4), 2)\}$. We define the group homomorphism $\varphi : \mathcal{F}(T) \to G$ by setting $\varphi(x) = ((\,), 0)$, $\varphi(y) = ((1\,3), 1)$, and $\varphi(z) = ((1\,2\,3\,4), 2)$. Compute $\varphi(zyxx)$.

Answer:

6. What is the inverse of $(4, 5)$ in the Cartesian product $\mathbb{Z}_5 \times U(7)$?

Answer:

7. Give the smallest non-negative integer which is a solution to the following system of congruences:

$$\begin{cases} x \equiv 5 \pmod{11} \\ x \equiv 3 \pmod{5} \end{cases}$$

Answer:

8. Determine the index of the kernel of the group homomorphism

$$\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}_2)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \pmod 2 & b \pmod 2 \\ c \pmod 2 & d \pmod 2 \end{pmatrix}$$

Answer:

9. List all the abelian groups of order 12 up to isomorphism (only one group per isomorphism class).

Answer:

10. Recall the presentation of the dihedral group $D_4 = \langle r, s \mid r^4, s^2, srsr \rangle$. What is the order of the quotient group $D_4/\langle r \rangle$?

Answer:

11. If $\sigma = (1\,2\,3)(3\,4\,5)(2\,3\,4)(1\,2\,3)$ is a permutation in S_6, and $\tau = \sigma^{-1}$, compute $\tau(1)$.

Answer:

12. Let $p(x) = 2x^2 + 2x + 12$ and $q(x) = 2x^2 - 6x + 4$ be polynomials in $\mathbb{Q}[x]$. Determine the (monic) $\gcd(p(x), q(x))$.

Answer:

13. The element $x \in \mathbb{Q}[x]$ is ...

Answer:

A a unit, but not a zero-divisor.

B a zero-divisor, but not a unit.

C both a unit and a zero-divisor.

D neither a unit, nor a zero-divisor.

14. Consider the polynomial $q(x) = x^3 + x + 1$ in $\mathbb{Z}_2[x]$. How many roots does it have (in $\mathbb{Z}_2$)?

Answer:

15. Which of these statements is **not always true** for a ring R and an ideal I of R?

A If I is prime, then R/I is an integral domain.

B If I is maximal, then R/I is an integral domain.

C If I is prime, then R/I is a field.

D If I is maximal, then R/I is a field.

Answer:

Questions with Explanation **[12 points]**

Write your answer in the boxes provided on this answer sheet at the end of the questionnaire.
Each proof step needs to be properly explained.

1. Let $(G, *)$ be an abelian group. Prove that, for any two subgroups H and K of G, the set

$$HK = \{h * k \mid h \in H,\ k \in K\}$$

is a subgroup of G. Clearly motivate each step.
Given a nonabelian group G and a subgroup $H \leq G$, is it always possible to find a subgroup $K \leq G$ such that HK is a subgroup of G?

2. Consider the four-element subgroup G of $\mathrm{Aut}(\mathbb{Z}_6)$ defined by

$$G = \left\{ \begin{array}{rl} f\colon \mathbb{Z}_6 \to \mathbb{Z}_6 \\ z \mapsto \alpha z + \beta \end{array} \;\middle|\; \alpha \in \{1,5\},\ \beta \in \{0,3\} \right\}.$$

G acts on $\mathbb{Z}_6$ by function application.

(a) List and compute all the orbits of this action.

(b) For each element of $\mathbb{Z}_6$, find its stabilizer subgroup.

(c) Find the fixed point set of the element $(z \mapsto 5z)$ of G.

3. Consider the ring

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

which is a subring of the ring of (2×2)-matrices with entries in $\mathbb{Z}_2$.
Let $\varphi : \mathbb{Z}_2[x] \to R$ be the unique ring homomorphism with

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \varphi(x) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Prove or disprove that the kernel of φ is a maximal ideal.

Answer to **Question 1.**

Answer to **Question 2.**

Answer to **Question 3.**

Additional writing space. *Clearly refer to this space in the original answer*

Additional writing space. *Clearly refer to this space in the original answer*