Course	:	**Algebra M7**	Name:
Date	:	Sample EXAM	
Time	:	2.5 h	Student number:

The use of electronic devices is not allowed.

The total number of points is 54. The grade for x points is $\frac{x+6}{6}$ rounded.

Final Answer

Each question is worth 3 points. Only a completely correct answer gains points.

1. What is the discrete logarithm of $(3\,6)(4\,5)$ with basis $(2\,1\,7)(3\,4\,6\,5)$ in S_7?

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2. What is the word of shortest length equivalent to $zxyzxyyx$ in the group with presentation $\langle x, y, z \mid xx, yz, zz, yy\rangle$?

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3. List the generators of the cyclic group generated by $(1\,8\,9\,2)$ in S_9.

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4. Let $\mathcal{F}(T)$ be the free group on $T = \{x, y, z\}$, and let G be the subgroup of the direct product $S_4 \times \mathbb{Z}_3$ generated by $\{((1\,2), 1), ((1\,3), 1), ((1\,2\,3\,4), 2)\}$. Furthermore, we define the group homomorphism $\phi\colon \mathcal{F}(T) \to G$ by setting $\phi(x) = ((1), 0)$, $\phi(y) = ((1\,3), 1)$, $\phi(z) = ((1\,2\,3\,4), 2)$. Which element of $S_4 \times \mathbb{Z}_3$ is $\phi(zyxx)$?

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5. What is the inverse of $(4, 5)$ in the direct product $\mathbb{Z}_5 \times U(7)$?

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6. Give the smallest non-negative integer which is a solution to the following system of congruences:

$$x \equiv 5 \pmod{11}$$
$$x \equiv 3 \pmod{5}$$

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7. What is the index of the kernel of the group homomorphism

$$\psi\colon SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_2)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \pmod 2 & b \pmod 2 \\ c \pmod 2 & d \pmod 2 \end{pmatrix}$$

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8. Mark all valid possibilities for which a field of characteristic 3 exists with this order.

   Not selecting all the correct options or also wrong options yields **0** points for the question.

   - 1 .... ☐
   - ∞ .... ☐
   - 9 .... ☐
   - 27 .... ☐

Mixed Multiple Choice

You can get 3 points per question. For each of the statements write 1 for true and 0 for false. If at least one of the three statements is not correctly recognized as true or false or if the answer is missing, then you get **0** points for the question.

1. (a) There is exaclty 1 abelian groups with 9 elements up to isomorphism.
   (b) $U(6)$ is isomorphic to $U(10)$.
   (c) In a group, each element has a unique inverse.

   (a) .... ______
   (b) .... ______
   (c) .... ______

2. (a) $2+\sqrt{5}$ is an element of $\mathbb{Q}(\sqrt[4]{5})$.
   (b) It is possible to construct $\sqrt[3]{5}$ with straightedge and compass.
   (c) There exists a group of order 16 which has a subgroup with 2 elements.

   (a) .... ______
   (b) .... ______
   (c) .... ______

3. (a) The function mapping all elements to 2 is a zero divisor in the ring of functions $\{f \mid f\colon \{1,2,3\} \to \mathbb{Z}_3\}$ with component-wise addition and multiplication
   (b) There is a ring homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ whose image is $2\mathbb{Z}$.
   (c) $\mathbb{R}$ is algebraically closed.

   (a) .... ______
   (b) .... ______
   (c) .... ______

Questions With Explanations

Write the answers on exam paper.

1. (6 points)

   Let
   $$G = \left\{ \begin{array}{rl} \phi\colon \mathbb{Z}_6 & \to \mathbb{Z}_6 \\ z & \mapsto \alpha z + \beta \end{array} \;\middle|\; \alpha \in \{1,5\}, \beta \in \{0,3\} \right\}$$
   be a subgroup of the group of bijective functions on $\mathbb{Z}_6$. Prove or disprove that G is commutative.

   Note that the group G acts on $\mathbb{Z}_6$. List the orbits of this action. Give the stabilizer subgroup of $3 \in \mathbb{Z}_6$.

2. (5 points)

   Let $p(x) = x^4+x^2+x+1 \in \mathbb{Z}_2[x]$. Compute the multiplicative inverse of $h(x) = x^3+x+1$ in $\mathbb{Z}_2[x]/\langle p(x)\rangle$ and prove that it is indeed an inverse.

3. (5 points)

   Consider the ring formed by the set
   $$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$
   a subring of the ring of (2×2)-matrices with entries in $\mathbb{Z}_2$.

   Let $\phi\colon \mathbb{Z}_2[x] \to S$ be the unique ring homomorphism with
   $$\phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \phi(x) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$
   Prove that the kernel of ϕ is not a maximal ideal.

4. (5 points)

   Determine the unique monic generator of the smallest ideal I of $\mathbb{Q}[x]$ that contains $p(x) = -x^2 + 2x + 3$ and $q(x) = -x^2 + x + 6$, namely $I = \{p(x)a(x) + q(x)b(x) \mid a(x), b(x) \in \mathbb{Q}[x]\}$, with the Euclidean algorithm.