Kenmerk: EWI2021/TW/MOR/MU/Mod7/Exam1

Exam 1, Module 7, Codes 202001360 & 202001364
Discrete Structures & Efficient Algorithms

Answers to questions 1-8 need to be motivated, arguments and proofs must be complete. No calculators. You are allowed to use a handwritten cheat sheet (A4, both sides) per topic (ADS, DM) during the exam.

For information: This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	ca. 1h	(30 points)
Discrete Mathematics (DM)	ca. 2h	(60 points)

The total is 30+60=90 points. Your grade is $1 + 0.1x$, x being the number of points, rounded to one digit. That means, you need 45 points to get a 5.5.

Please use a new sheet of paper for each part (ADS, DM), as the ADS and DM parts will be corrected separately!

Algorithms & Data Structures

See other pdf's for example questions for the ADS part.

Discrete Mathematics

4. (10 points)

   (a) Show that the Diophantine equation $1000s + 444t = 2$ has no solution for $s, t \in \mathbb{Z}$.

   (b) Show that for $a, b, c \in \mathbb{Z}$, if gcd$(a, b)|c$, then equation $as + bt = c$ has a solution in integers (s, t).

5. (10 points)

   (a) Compute the solution to the recurrence relation

$$a_n - 10a_{n-1} + 21a_{n-2} = 60 \cdot 3^n \quad (n \geq 2) \quad \text{with} \quad a_0 = 2 \text{ en } a_1 = -5\,.$$

(b) Consider strings in $\{0, 1, 2\}^*$. Let a_n be the number of strings in $\{0, 1, 2\}^*$ of length n that do not contain the substring 01 and neither 02. Compute a_1, a_2, a_3 and a recurrence relation for a_n for all $n \geq 4$. (You do not need to solve this recurrence relation.)

6. (10 points) Let $G = (V, E)$ be a simple, undirected graph, with edge weights $d_e > 0$, $e \in E$. Let $T \subseteq E$ be a minimum weight spanning tree (MST) for G. Moreover, for a fixed node $s \in V$, let us denote by $D(s, v)$ all edges that lie on shortest (s, v)-paths, for $v \in V$. Let $D(s) := \bigcup_{v \in V} D(s, v)$. Prove that $T \cap D(s) \neq \emptyset$.

7. (9 points) Let $G = (V, E)$ be a simple, bipartite undirected graph. Let $|V| = n$ and $|E| = m > 1$. Prove or give a counterexample:

   (a) If $m \leq 2n - 4$, Then G is planar.

   (b) If G is planar, then $m \leq 2n - 4$.

8. (9 points) How many possibilities are there to distribute 50€ over three persons, such that nobody gets less than 10€, and somebody gets at most 15€? Use a generating function to compute the answer.

9. (3 points each) For each of the following four claims, decide if true or false or if you would rather not give an answer. A correct answer gives **3**, an incorrect answer $-\mathbf{2}$ and not giving an answer **0 points** (minimum total number of points for Question 9 is 0 points). **Instead of guessing, it may be better not giving an answer.**

   (a) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$, $e \in E$. Then any two minimum spanning trees T_1 and T_2 for G must have a nonempty intersection, that is, $T_1 \cap T_2 \neq \emptyset$.

   True . . . . ☐
   False . . . . ☐
   I prefer to not give an answer . . . . ☐

   (b) Consider a capacitated network $G = (V, A, c)$, where V is the set of vertices, A is the set of directed arcs, and $c_a \geq 0$, $a \in A$ are the arc capacities. Then there always exists a maximum flow f_a, $a \in A$, such that either $f_a = 0$ or $f_a = c_a$ for all $a \in A$.

   True . . . . ☐
   False . . . . ☐
   I prefer to not give an answer . . . . ☐

(c) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$ such that $w_e \neq w_{e'}$ for all $e, e' \in E$, $e \neq e'$. Let $s \in V$ be fixed. Then for all $v \in V$ there is a unique shortest (s, v)-path.

True . . . . ☐
False . . . . ☐
I prefer to not give an answer . . . . ☐

(d) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$ such that $w_e \neq w_{e'}$ for all $e, e' \in E$, $e \neq e'$. Then there is a unique minimum spanning tree T.

True . . . . ☐
False . . . . ☐
I prefer to not give an answer . . . . ☐