Solutions

Ex 4

a) $\begin{bmatrix} 141 & 1 & 0 \\ 34 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 1 & -4 \\ 34 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 1 & -4 \\ 4 & -6 & 25 \end{bmatrix} \sim$

last non-zero remainder

$\begin{bmatrix} 1 & 7 & -29 \\ 4 & -6 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & -29 \\ 0 & -34 & 141 \end{bmatrix}$

Therefore $1 = 7 \cdot 141 - 29 \cdot 34$

$\Leftrightarrow 30 = 7 \cdot 30 \cdot 141 - 29 \cdot 30 \cdot 34$

$= \underbrace{210}_{x} \cdot 141 \underbrace{- 870}_{y} \cdot 34$

b) $\gcd(a,b) = 1$, therefore there exist $x, y \in \mathbb{Z}$ such that

$x \cdot a + y \cdot b = 1$.

Multiplying with c, we get

$a \cdot x \cdot c + bcy = c$

Now, a divides the first term, and a divides the second term, so a divides their sum: $a | c$. ∎

c) This was a tutorial exercise. By induction:

<u>base:</u> $\gcd(F_0, F_1) = 1$

<u>hypothesis:</u> Assume $\gcd(F_n, F_{n-1}) = 1$

<u>inductive step:</u> $\gcd(F_{n+1}, F_n) =$

$= \gcd(F_n + F_{n-1}, F_n)$

Assume $\exists\, d > 1$ such that $d | (F_n + F_{n-1})$ and $d | F_n$.

Then $d | (F_n + F_{n-1} - F_n) \Rightarrow d | F_{n-1}$ } contradicts inductive hypothesis.

∎

Ex 5/ Solution 1: Let x be the number of vertices of T with degree 1. Then

$$2m = \sum_{v \in V} \deg(v) \geq k + (n-x-1)\cdot 2 + x\cdot 1$$
$$= k + 2n - 2x - 2 + x$$
$$= k + 2n - 2 - x.$$

Note that in a tree $m = n-1$. Therefore

$$2n-2 \geq k + 2n - 2 - x$$

$$\Leftrightarrow x \geq k. \quad \blacksquare$$

Solution 2: Let $v \in V$ such that $\deg(v) \geq k$.
Removing v partitions T into at least k components.
By the hint each of these components has at least two leaves, at most one of them could be connected to v. So T has at least k leaves. $\blacksquare$

Ex 6/ (From tutorial sessions and also true/false in sample test)

- Assume that there exist two different MST's T_1 and T_2.
- Then $\exists\ e = \{u,v\} \in T_1 \setminus T_2$.
- Let $P_{T_2}(u,v)$ be the unique path between u and v in T_2.
- By path-condition in T_2: $c(f) \leq c(e)$ for all edges $f \in P_{T_2}(u,v)$, and since all costs are different actually $c(f) < c(e)$ for all $f \in P_{T_2}(u,v)$.

- But this means that in the cut induced by $T_1 \setminus e$ there exists at least one edge $f \in P_{T_2}(u,v)$

with $c(f) < c(e)$. So by the cut-condition, T_1 is not a minimum spanning tree. Contradiction ■

Ex 7

Algorithm & Proof follow exactly the proof of strong duality from the lecture.

Algorithm:

- Compute residual graph $G_f = (V', E')$:
  for all $v \in V$: create a copy $v' \in V'$;
  for all $e = (u,v) \in E$:
  $r_f(u,v) = c(u,v) - f(u,v)$;
  $r_f(v,u) = f(u,v)$
- Run BFS from s in G_f, & Let S' be the set of vertices reached & $T := V' \setminus S'$
- Return cut $[S, T]$

Running Time

- Computing G_f $\in O(m+n)$
- BFS $\in O(m+n)$ because $|E'| \leq 2|E|$ and $|V'| = |V|$.

Correctness: It suffices to show the following claim:

$\boxed{\text{All edges in } E \cap \delta(S) \text{ are saturated by } f.}$

The result then follows by the max-flow min-cut theorem.

Proof of claim: Consider any $(u,v) \in E$:
- if $(u,v) \in (S,T) \Rightarrow f(u,v) = c(u,v)$
  (otherwise $(u,v) \in E'$ and $v \in S$)
- if $(u,v) \in (T,S) \Rightarrow f(u,v) = 0$
  (otherwise $(v,u) \in E'$ and $u \in S$) ∎

Ex. 8

(a) (Adapted from lecture: Let the seating positions be numbered from 1 to n. Then we are looking for the number of subsets of $S = \{1, \dots n\}$ that do not contain consecutive numbers.)

Let a_n be the number of such arg.

(Base Case) • For $n = 0$, $S = \emptyset$ and $a_0 = 1 = F_2$ many seating arrangements.

• For $n = 1$, $S = \{1\}$ and $a_1 = 2 = F_3$ many seating arrangements.

(Hypothesis). Assume that the number of arrangements is $a_k = F_{k+2}$ for $k < n$

(Step). For n, let $S' \subseteq S$ be a valid arrangement. Then

- Case 1: $n \in S' \Rightarrow S' \setminus \{n\} \subseteq \{1, \dots n-2\}$
  $\Rightarrow a_{n-2}$ such sets $S' \setminus \{n\}$.

- Case 2: $n \notin S' \Rightarrow S' \subseteq \{1, \dots, n-1\}$
  $\Rightarrow a_{n-1}$ such sets S'
- In total therefore

$$a_n = a_{n-1} + a_{n-2}$$

$$\underset{\text{by hypothesis}}{=} F_{n+1} + F_n = F_{n+2} \quad \blacksquare$$

(b) Analogous to the step in (a).
Let a_n be the number of valid arrangements
Pick an arbitrary seat i and let S'
be a valid arrangement. Then:

- Case 1: $i \in S'$. Then the two neigbours of i cannot be in S' and the problem reduces to that with $n-3$ seats placed in a line. So, by (a) there are F_{n-1} such arrangements.
- Case 2: $i \in S'$. Then the problem reduces to that with $n-1$ seats placed in a line. So by (a) there are F_{n+1} such arrangements.

In total $a_n = F_{n-1} + F_{n+1}$.

(c) Characteristic Polynomial:

$$\underbrace{x^2-7+10}_{(x-5)(x-2)}=0$$

So $\lambda_1 = 5$, $\lambda_2 = 2$.

General Solution: $a_n = C_1 \cdot 5^n + C_2 \cdot 2^n$
and by using the initial values:

- $n=0$. $a_0 = 0 = C_1 + C_2 \Rightarrow C_1 = -C_2$.
- $n=1$ $a_1 = 3 = 5C_1 + 2C_2 = -3C_2$

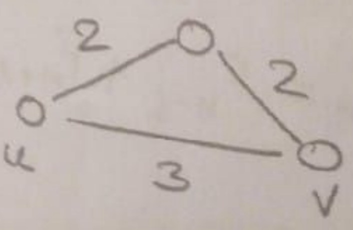
Thus $C_2 = -1$, $C_1 = 1$, and

$$a_n = 5^n - 2^n.$$

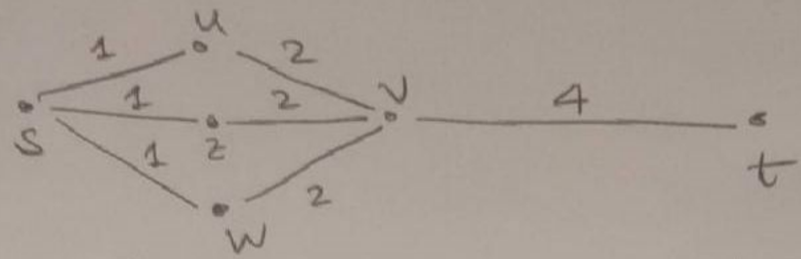
Ex 9

(a) True. We know that a graph has an Euler tour iff all vertices have even degree. In a complete bipartite graph the degree of each vertex is the "other" set in the bipartition and the statement follow.

(b) False. We have seen this repeatedly in the lecture. Here is a counterexample:



(c) False. Counterexample



when adding $k=1$ to the capacity of every edge the minimum s-t cut changes.

(d) False. The graph is homeomorphic to $K_{3,3}$.

(e) True. We know that

$$g(y) = \frac{1}{1-y} \text{ generates } 1, 1, 1, \dots$$

and therefore $g(y) = 1 + y + y^2 + \dots$

In turn $f(x) = g(x^2) = 1 + x^2 + x^4 + \dots$

(f) False.

The coefficient of x^{50} in $(x^7 + x^8 + \dots)^6$ is the same as the coefficient of x^8 in $(1 + x + x^2 + \dots)^6$. From the lecture, that is $\binom{8+6-1}{8} = \binom{13}{8} \neq \binom{14}{8}$.