Course : Linear Algebra
Module : 2A
Course code : 202001205

Date : 17 April 2025
Time : 13:45 – 15:45
Reference : IEM & TCS

A

Linear Algebra

Exam

Instructions

This exam contains 10 questions. You shall use the attached *answer form* to submit your answers.

- For questions 1–4, you are only required to fill in the **final answer** on the answer form.
- For questions 5–10, you are required to write down a **full calculation and argumentation**.

You will hand in your answer form only. If you run out of space, you can use the extra space at the end of the answer form. Refer clearly to that space in the original answer.
Do not write with red pen or pencil. Do not use correction fluid or tape.
The use of electronic devices is not allowed!

Final answer questions

Write only your final answer on the answer form.

1. For each of the following statements, determine whether it is true or false. [3 pt]

(a) Let A be a square matrix such that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$. Then A is invertible.

(b) Let A be a square matrix. If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A, then $\mathbf{u} + \mathbf{v}$ is also an eigenvector of A.

(c) For every two matrices A and B of size 2×2 we have that $\det(A+B) = \det(A) + \det(B)$.

(d) Let $A\mathbf{x} = \mathbf{b}$ be a linear system that has no solutions. Then the reduced echelon form of A must have a zero row.

2. Let A be the matrix given by $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$.

(a) Find a basis for $\operatorname{Col} A$. [2 pt]

(b) Determine the dimension of $\operatorname{Col} A$. [1 pt]

3. The augmented matrix of a linear system $A\mathbf{x} = \mathbf{b}$ and a vector $\mathbf{v}$ are given below

$$\left(A \mid \mathbf{b}\right) = \left(\begin{array}{cccc|c} a^2-1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & a+3 & a+4 & 1 \\ 0 & 0 & 0 & a+4 & 1 \end{array}\right), \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Find all $a \in \mathbb{R}$ such that the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. [2 pt]

(b) Find all $a \in \mathbb{R}$ such that the linear system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. [2 pt]

(c) Find all $a \in \mathbb{R}$ such that the linear system $A\mathbf{x} = \mathbf{b}$ has the vector $\mathbf{v}$ as a solution. [1 pt]

4. Find all matrices X and Y of size 3×3 that satisfy the following system of matrix equations [2 pt]

$$\begin{cases} X - Y = 5I \\ X + Y = I \end{cases}$$

Continues on the following page.

Open questions

The full solutions to questions 5–10 must be clearly written down on the answer form, including calculations and argumentations.

Points will not be awarded for reaching a correct result if this is not supported by a correct procedure and by a sound and clear argumentation.

5. (a) Let $\alpha, \beta, \gamma \in \mathbb{R}$. Show that $\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+\alpha & 1 & 1 \\ 1 & 1 & 1+\beta & 1 \\ 1 & 1 & 1 & 1+\gamma \end{pmatrix} = \alpha\beta\gamma.$ [2 pt]

(b) Let A be a 4×4 matrix such that [2 pt]

$$A^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

Show that $\mathbf{0}$ is the only solution to the system $A\mathbf{x} = \mathbf{0}$.
(Hint: You can use part (a) to compute $\det(A^2)$).

6. Determine the matrix $(AA^T)^{-1}A$, where the matrix A and its inverse A^{-1} are given by: [3 pt]

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 2 & 1 & -3 \\ -2 & -1 & 4 \\ -1 & 0 & 1 \end{pmatrix}.$$

7. Find $x_1, x_2, x_3 \in \mathbb{R}$ such that the matrix $A = \begin{pmatrix} 2 & x_1 \\ x_2 & x_3 \end{pmatrix}$ has 1 as an eigenvalue and such that [4 pt]

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}.$$

8. (a) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by an angle of θ radians around the origin. Find $a, b \in \mathbb{R}$ if the representation matrix of T_1 is $\begin{pmatrix} \frac{1}{2} & a \\ \frac{\sqrt{3}}{2} & b \end{pmatrix}$. [2 pt]

(b) Let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the x_1-axis and T_1 be as in part (a). Find the representation matrix of $T_1 \circ T_2^{-1}$. [2 pt]

9. Let A, B be two 3×3 matrices.

(a) Let $\mathbf{w}$ be an eigenvector of both A and B. Show that $\mathbf{w}$ is in $\operatorname{Null}(AB - BA)$. [2 pt]

(b) Assume that $AB = BA$, let λ be an eigenvalue of B, and let $E_\lambda(B)$ be the eigenspace of B associated with λ. Prove that if a vector $\mathbf{v}$ is in $E_\lambda(B)$, then $A\mathbf{v}$ is in $E_\lambda(B)$. [2 pt]

10. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of linearly independent vectors in $\mathbb{R}^4$ and let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation.

(a) Show that if $\mathbf{v} \in \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then $T(\mathbf{v}) \in \operatorname{Span}\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$. [2 pt]

(b) Assume that T is one-to-one. Prove that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent. [2 pt]

Total: 36 pt