

LINEAR ALGEBRA SAMPLE TEST 1 SOLUTIONS

1.
 - i **False.** It may be inconsistent.
 - ii **False.** It may happen that the reduced echelon form of $(A \mid b)$ and the reduced echelon form of A both have a pivot in the last column. In that case the linear system will have no solution.
 - iii **False.** This is true only when $\mathbf{b} = \mathbf{0}$. It is false in general.
 - iv **False.** It may have no solutions.
2. Using row operations we find the following forms

$$\left(\begin{array}{ccc|c} 0 & 5 & -5 & b_1 \\ 3 & 1 & 8 & b_2 \\ -3 & -4 & -5 & b_3 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 1 & 8 & b_2 \\ 0 & 5 & -5 & b_1 \\ -3 & -4 & -5 & b_3 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 + R_1} \left(\begin{array}{ccc|c} 3 & 1 & 8 & b_2 \\ 0 & 5 & -5 & b_1 \\ 0 & -3 & 3 & b_2 + b_3 \end{array} \right)$$

$$\xrightarrow{R_2 \rightarrow \frac{1}{5}R_2} \left(\begin{array}{ccc|c} 3 & 1 & 8 & b_2 \\ 0 & 1 & -1 & \frac{b_1}{5} \\ 0 & -3 & 3 & b_2 + b_3 \end{array} \right) \xrightarrow{R_3 \rightarrow 3R_2 + R_3} \left(\begin{array}{ccc|c} 3 & 1 & 8 & b_2 \\ 0 & 1 & -1 & \frac{b_1}{5} \\ 0 & 0 & 0 & 3b_1/5 + b_2 + b_3 \end{array} \right)$$

Therefore, the system is consistent when $3b_1 + 5b_2 + 5b_3 = 0$, i.e., when $b_1 = -\frac{5}{3}(b_2 + b_3)$.

3. We first simplify the expression into

$$(A^T)^T - B^T + C(C^{-1}B) = A - B^T + IB.$$

Because B is symmetric, we have :

$$A - B^T + IB = A - B + B = A.$$

4. We perform row operations so that from $(B|I)$ we get $(I|B^{-1})$. If you do all computations correctly you will find the following inverse matrix:

$$\begin{pmatrix} -1 & -1 & 1 \\ 2 & 2 & -1 \\ 2 & 3 & -1 \end{pmatrix}$$

5. We first compute the reduced echelon form of A :

$$\left(\begin{array}{cccc} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{array} \right) \xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 - R_1} \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & 1 \\ 1 & 1 & 6 & 3 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 - R_1}$$

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -1 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 + R_2} \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \left(\begin{array}{cccc} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Since the first two columns of the row reduced echelon form are pivot columns, we obtain that

a basis for $\text{Col } A$ is $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \right\}$.

6. The given vectors are linearly independent exactly when the linear system

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} \alpha \\ 1 \\ -\alpha \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 2\alpha \\ 3\alpha + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

only has the zero solution. Using the augmented matrix we have

$$\left(\begin{array}{ccc|c} 1 & \alpha & 1 & 0 \\ 0 & 1 & 2\alpha & 0 \\ 0 & -\alpha & 3\alpha + 1 & 0 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 + \alpha R_2} \left(\begin{array}{ccc|c} 1 & \alpha & 1 & 0 \\ 0 & 1 & 2\alpha & 0 \\ 0 & 0 & 2\alpha^2 + 3\alpha + 1 & 0 \end{array} \right)$$

From this we see that when $2\alpha^2 + 3\alpha + 1 \neq 0$, the system has no free variables and, hence, it can only have $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as a solution. Therefore, the given vectors are linearly independent when $2\alpha^2 + 3\alpha + 1 \neq 0$, i.e., when $\alpha \neq -1$ and $\alpha \neq -\frac{1}{2}$.

Alternative Solution: We know that

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1 \\ -\alpha \end{pmatrix}, \begin{pmatrix} 1 \\ 2\alpha \\ 3\alpha + 1 \end{pmatrix} \right\}$$

is linearly independent if and only if

$$\det \begin{pmatrix} 1 & \alpha & 1 \\ 0 & 1 & 2\alpha \\ 0 & -\alpha & 3\alpha + 1 \end{pmatrix} \neq 0.$$

We can compute the determinant

$$\det \begin{pmatrix} 1 & \alpha & 1 \\ 0 & 1 & 2\alpha \\ 0 & -\alpha & 3\alpha + 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 2\alpha \\ -\alpha & 3\alpha + 1 \end{pmatrix} = 2\alpha^2 + 3\alpha + 1.$$

Therefore, the given vectors are linearly independent when $2\alpha^2 + 3\alpha + 1 \neq 0$, i.e., when $\alpha \neq -1$ and $\alpha \neq -\frac{1}{2}$.

7. If we take the initial matrix as:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

and perform the following elementary row operations:

- $R_1 \rightarrow xR_1$,
- $R_2 \rightarrow xR_2$,
- $R_2 \rightarrow R_2 - R_1$,
- $R_1 \rightarrow R_1 + R_3$,

then we get the matrix:

$$B = \begin{pmatrix} ax + g & bx + h & cx + i \\ x(d - a) & x(e - b) & x(f - c) \\ g & h & i \end{pmatrix}$$

Only the first and the second of the elementary row operations defined above change the determinant (each as the product of x).

Therefore we can see that $\det B = x^2 \det A$.

8. Let λ be an eigenvalue of A , we need to show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$. Since λ is an eigenvalue of A , there exists a non-zero vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$. Moreover,

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$$

and

$$A^3\mathbf{v} = A(A^2\mathbf{v}) = A(\lambda^2\mathbf{v}) = \lambda^2 A\mathbf{v} = \lambda^2 \lambda \mathbf{v} = \lambda^3 \mathbf{v}$$

Therefore, we have

$$(A^3 - 2A + 3I)\mathbf{v} = A^3\mathbf{v} - 2A\mathbf{v} + 3I\mathbf{v} = \lambda^3\mathbf{v} - 2\lambda\mathbf{v} + 3\mathbf{v} = (\lambda^3 - 2\lambda + 3)\mathbf{v}$$

This proves that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$ (with corresponding eigenvector \mathbf{v}).

9. Let $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix}$. Then we know that

$$A\mathbf{v}_1 = -\mathbf{v}_1, \quad A\mathbf{v}_2 = 0\mathbf{v}_2 = \mathbf{0}, \quad \text{and} \quad A\mathbf{v}_3 = \mathbf{v}_3 \quad (1)$$

Moreover, since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors corresponding to distinct eigenvalues, we know that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Therefore, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

The idea is to express the vector $\begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and then apply

A^{10} to this linear combination. We first find all $x_1, x_2, x_3 \in \mathbb{R}$ that satisfy the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix}.$$

The above equation is exactly the following linear system:

$$\begin{cases} -3x_1 - 2x_2 - 4x_3 = -3 \\ x_1 + x_2 + 2x_3 = 2 \\ 5x_1 + 4x_2 + 7x_3 = 16 \end{cases}$$

Using the augmented matrix we have

$$\left(\begin{array}{ccc|c} -3 & -2 & -4 & -3 \\ 1 & 1 & 2 & 2 \\ 5 & 4 & 7 & 16 \end{array} \right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & -9 \end{array} \right)$$

(make sure you can do all the necessary row operations). Therefore, we have that

$$\begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix} = -\mathbf{v}_1 + 21\mathbf{v}_2 - 9\mathbf{v}_3.$$

Thus

$$A \begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix} = A(-\mathbf{v}_1 + 21\mathbf{v}_2 - 9\mathbf{v}_3) = -A\mathbf{v}_1 + 21A\mathbf{v}_2 - 9A\mathbf{v}_3 = \mathbf{v}_1 - 9\mathbf{v}_3,$$

where the last equality follows from the Equations of (1). Similarly,

$$A^2 \begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix} = A(\mathbf{v}_1 - 9\mathbf{v}_3) = A\mathbf{v}_1 - 9A\mathbf{v}_3 = -\mathbf{v}_1 - 9\mathbf{v}_3.$$

If we continue in this way we find,

$$A^{10} \begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix} = -(-1)^{10}\mathbf{v}_1 - 9\mathbf{v}_3 = -\mathbf{v}_1 - 9\mathbf{v}_3 = - \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} - 9 \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 39 \\ -19 \\ -68 \end{pmatrix}$$

Alternative Solution: If we find matrices P, D, P^{-1} such that $A = PDP^{-1}$, then we know that $A^{10} = PD^{10}P^{-1}$. Therefore, $A^{10}\mathbf{v} = PD^{10}P^{-1}\mathbf{v}$.

We know from theory that

$$P = \begin{pmatrix} -3 & -2 & -4 \\ 1 & 1 & 2 \\ 5 & 4 & 7 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We now find P^{-1} :

$$\left(\begin{array}{ccc|ccc} -3 & -2 & -4 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 5 & 4 & 7 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\text{row operations}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right)$$

(make sure you can do all the necessary row operations).

Therefore,

$$P^{-1} = \begin{pmatrix} -1 & -2 & 0 \\ 3 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

Moreover, we have that

$$D^{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus we find

$$A^{10}\mathbf{v} = PD^{10}P^{-1}\mathbf{v} = \begin{pmatrix} 7 & -2 & 4 \\ -3 & 2 & -2 \\ -12 & 4 & -7 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix} = \begin{pmatrix} 39 \\ -19 \\ -68 \end{pmatrix}.$$

10. We need to write the vectors \mathbf{v}_1 and \mathbf{v}_2 as a linear combination of the given vector,

$$\begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

We can solve the system and find that $c_1 = -2$ and $c_2 = 3$. Therefore,

$$\begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

Since the transformation T is linear we have

$$T \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = -2T \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + 3T \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Therefore,

$$T \begin{pmatrix} 5 \\ 6 \\ 13 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \end{pmatrix}$$

11. Given that $\mathbf{y}_1, \mathbf{y}_2$ are both in the image of T , we need to show that for all $a, b \in \mathbb{R}$ we have that $a\mathbf{y}_1 + b\mathbf{y}_2$ is also in $\text{im}(T)$.

Since \mathbf{y}_1 belongs to $\text{im}(T)$ we know that there exists $\mathbf{x}_1 \in \mathbb{R}^n$ such that $T(\mathbf{x}_1) = \mathbf{y}_1$. Similarly, there exists $\mathbf{x}_2 \in \mathbb{R}^n$ such that $T(\mathbf{x}_2) = \mathbf{y}_2$. Then

$$a\mathbf{y}_1 + b\mathbf{y}_2 = aT(\mathbf{x}_1) + bT(\mathbf{x}_2) = T(a\mathbf{x}_1) + T(b\mathbf{x}_2) = T(a\mathbf{x}_1 + b\mathbf{x}_2)$$

Thus, $a\mathbf{y}_1 + b\mathbf{y}_2$ is in $\text{im}(T)$.