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LINEAR ALGEBRA $_A$ Date : April 02, 2024
Time : 13.45 – 15.45 hrs

First read these instructions carefully:

This test contains 11 exercises. The complete solutions of Exercises 2, 5, 6, 7, 8, 9, and 11 must be accurately written down including calculations and argumentation and for the other exercises you are only required to fill in the final answers on the answer sheet at the end of this test. You must hand in the answer sheet.
The use of dictionaries or electronic devices is not allowed.
The grade is determined by dividing the total number of points by 4 and adding 1.

1. (3 pts.) *Fill in your final answer to this exercise on the supplied answer sheet. Each error results in a one-point deduction from the total score (3 pts).*

   Determine whether each of the following statements is true or false.

   i. A linear system of five equations in four unknowns is always consistent.

   ii. Consider the linear system $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{R}^{n\times n}$. If the reduced echelon form of the coefficient matrix A has a pivot in the last column then the linear system has a unique solution.

   iii. For a linear system $A\mathbf{x} = \mathbf{b}$, if $\mathbf{u}$ and $\mathbf{v}$ are two solutions, then $\mathbf{u} + \mathbf{v}$ is also a solution.

   iv. A linear system of four equations with five unknowns must have infinitely many solutions.

2. (3 pts.) *Use the supplied space and include clear argumentation and calculation.*

   Find all values of b_1, b_2, and b_3 for which the system $A\mathbf{x} = \mathbf{b}$ is consistent with
   $A = \begin{pmatrix} 0 & 5 & -5 \\ 3 & 1 & 8 \\ -3 & -4 & -5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

3. (3 pts.) *Fill in your final answer to this exercise on the supplied answer sheet.*

   Let A, B, C be the following matrices in $\mathbb{R}^{3\times 3}$

   $$A = \begin{pmatrix} 1 & 5 & 7 \\ 3 & 2 & 1 \\ 5 & 6 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 8 & -2 \\ 1 & -2 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 4 & 9 \\ -4 & 8 & 1 \end{pmatrix}.$$

   By first simplifying, compute the following expression:$(A^T - B)^T + C(B^{-1}C)^{-1}$.

4. (2 pts.) *Fill in your final answer to this exercise on the supplied answer sheet.*

   We are given the matrix

   $$B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

   Determine the inverse of the matrix B.

5. (4 pts.) *Use the supplied space and include clear argumentation and calculation.*

   Let $A \in \mathbb{R}^{3\times 4}$ be the following matrix:

   $$A = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{pmatrix}$$

   Determine a basis for Col A.

6. (3 pts.) *Use the supplied space and include clear argumentation and calculation.*

   Find all values of $\alpha \in \mathbb{R}$ such that the following set of vectors is linearly independent:

   $$\left\{ \begin{pmatrix} 1 \\ 2\alpha \\ 3\alpha + 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1 \\ -\alpha \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

7. (3 pts.) *Use the supplied space and include clear argumentation and calculation.*

   Show that the following equation holds for any values of $a, b, c, d, e, f, g, h, i$, and x in $\mathbb{R}$:

   $$\det \begin{pmatrix} ax + g & bx + h & cx + i \\ x(d - a) & x(e - b) & x(f - c) \\ g & h & i \end{pmatrix} = x^2 \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

8. (4 pts.) *Use the supplied space and include clear argumentation and calculation.*

   Suppose that λ is an eigenvalue of a matrix $A \in \mathbb{R}^{n\times n}$. Show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$.

9. (3 pts.) *Use the supplied space and include clear argumentation and calculation.*

   Let $A = \begin{pmatrix} 1 & -14 & 4 \\ -1 & 6 & -2 \\ -2 & 24 & -7 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix}$. Given the eigenvalues of A as -1, 0, and 1, as well as the below corresponding eigenspaces:

   $$E_{-1} = \text{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \right\}, E_0 = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \right\}, E_1 = \text{Span}\left\{ \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix} \right\}.$$

   Find $A^{10}\mathbf{v}$.

10. (4 pts.) *Fill in your final answer to this exercise on the supplied answer sheet.*

    For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ we have: $T \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$

    Find $T \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix}$.

11. (4 pts.) *Use the supplied space and include clear argumentation and calculation.*

    Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

    Given that $\mathbf{y_1}$ and $\mathbf{y_2}$ are both in the image of T, show that $a\mathbf{y_1} + b\mathbf{y_2}$ is also in im T for all $a, b \in \mathbb{R}$.