Final answer questions:

Question 1:

	Version A	Version B	Version C	Version D
(a)	T	T	F	T
(b)	T	F	T	F
(c)	F	T	T	F
(d)	F	F	F	T

Question 2:

	Version A	Version B	Version C	Version D
a =	0	-2	2	0
b =	3	1	-1	-3
c =	4	4	-2	-2

Question 3:

Version A:

(a) $Null(A) = Span\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\-1\\1\\0\end{pmatrix}\right\}$

(b) Basis: $\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\-1\\1\\0\end{pmatrix}\right\}$

(c) $\dim Null(A) = 2$

Version B:

(a) $Null(A) = Span\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\-1\\1\\0\end{pmatrix}\right\}$

(b) Basis: $\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\-1\\1\\0\end{pmatrix}\right\}$

(c) $\dim Null(A) = 2$

Version C:

(a) $Null(A) = Span\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(b) Basis $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(c) $\dim Null(A) = 2$

Version D:

(a) $Null(A) = Span\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(b) Basis: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(c) $\dim Null(A) = 2$

<u>Question 4:</u>

Version A: $X = \frac{2}{3}A$

Version B: $X = \frac{3}{2}A$

Version C: $X = \frac{3}{5}A$

Version D: $X = \frac{5}{3}A$

Question 5. *Write a full calculation/argumentation clearly in the box.*

Since $\mathrm{Null}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ we know that

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}2 & x_1\\ x_2 & x_3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases} 2 + x_1 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = -2 \\ x_2 = -x_3 \end{array}$$

Since $\mathrm{col}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ we find that there exists $c \in \mathbb{R}$ such that

$$\begin{pmatrix}2\\x_2\end{pmatrix} = c\begin{pmatrix}1\\2\end{pmatrix}$$

$$\Rightarrow \begin{cases} 2 = c \\ x_2 = 2c \end{cases}$$ Therefore, $x_2 = 2 \cdot 2 = 4$

This gives that $x_3 = -x_2 = -4$

Thus

$$A = \begin{pmatrix}2 & -2\\ 4 & -4\end{pmatrix}$$

Question 6. *Write a full calculation/argumentation clearly in the box.*

We have

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{1}{2} R_2}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_4 \to R_4 + \frac{1}{2} R_3} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_5 \to R_5 + \frac{1}{2} R_4}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Since the elementary row operations that we performed do not change the determinants involved we have

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} =$$

$$= 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

Question 7a. *Write a full calculation/argumentation clearly in the box.*

We need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\begin{pmatrix}0\\0\\1\end{pmatrix}+c_2\begin{pmatrix}1\\0\\-2\end{pmatrix}+c_3\begin{pmatrix}-3\\1\\6\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow \left\{\begin{array}{c} c_2-3c_3=1 \\ c_3=1 \\ c_1-2c_2+6c_3=1 \end{array}\right\} \Rightarrow \begin{array}{l} c_2=1+3c_3=1+3=4 \\ c_3=1 \\ c_1=1+2c_2-6c_3 \\ \quad =1+2\cdot4-6 \\ \quad =1+8-6=3 \end{array}$$

Therefore

$$[\vec{v}]_S=\begin{pmatrix}3\\4\\1\end{pmatrix}$$

Question 7b. *Write a full calculation/argumentation clearly in the box.*

We know

$$\vec{w}=-1\begin{pmatrix}0\\0\\1\end{pmatrix}+2\begin{pmatrix}1\\0\\-2\end{pmatrix}+1\begin{pmatrix}-3\\1\\6\end{pmatrix}$$

So

$$A\vec{w}=A\left(-1\begin{pmatrix}0\\0\\1\end{pmatrix}+2\begin{pmatrix}1\\0\\-2\end{pmatrix}+1\begin{pmatrix}-3\\1\\6\end{pmatrix}\right)$$

$$=-A\begin{pmatrix}0\\0\\1\end{pmatrix}+2A\begin{pmatrix}1\\0\\-2\end{pmatrix}+A\begin{pmatrix}-3\\1\\6\end{pmatrix}=$$

$$=-\frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}+2\,\frac{1}{2}\begin{pmatrix}1\\0\\-2\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-3\\1\\6\end{pmatrix}=\begin{pmatrix}0\\1/3\\-1/2\end{pmatrix}$$

Question 8a. *Write a full calculation/argumentation clearly in the box.*

Since $T_1\begin{pmatrix}2\\1\end{pmatrix} = \vec{e}_1$ we get $T_1^{-1}(\vec{e}_1) = \begin{pmatrix}2\\1\end{pmatrix}$

Since $T_1\begin{pmatrix}1\\1\end{pmatrix} = \vec{e}_2$ we get $T_1^{-1}(\vec{e}_2) = \begin{pmatrix}1\\1\end{pmatrix}$

Therefore, the representation matrix of T_1^{-1} is $\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$

Question 8b. *Write a full calculation/argumentation clearly in the box.*

T_2^{-1} is the counterclockwise rotation by $\pi/4$ radians around the origin. Therefore the representation matrix of T_2^{-1} is

$$\begin{pmatrix}\cos(\pi/4) & -\sin(\pi/4)\\ \sin(\pi/4) & \cos(\pi/4)\end{pmatrix} = \begin{pmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{pmatrix}$$

Therefore the representation matrix of T is:

$$\begin{pmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}\sqrt{2}/2 & 0\\ 3\sqrt{2}/2 & 2\sqrt{2}/2\end{pmatrix}$$

Question 8c. *Write a full calculation/argumentation clearly in the box.*

Since T is invertible we have that T is onto.

Question 9a. *Write a full calculation/argumentation clearly in the box.*

For all $\vec{v} \in \mathbb{R}^n$ we have that

$A((A+5I)\vec{v}) = [A(A+5I)]\vec{v} = 0\vec{v} = 0$

Therefore

$(A+5I)\vec{v} \in Null(A)$

Question 9b. *Write a full calculation/argumentation clearly in the box.*

We have that $A(A+5I)=0$

(Since A is invertible A^{-1} exists (and $A^{-1}A=I$))

Therefore $A^{-1} \cdot A(A+5I) = A^{-1} \cdot 0$

$\Rightarrow I \cdot (A+5I) = 0$

$\Rightarrow A+5I=0$

$\Rightarrow A=-5I$

Thus $\det(A) = \det(-5I) \underset{\substack{\uparrow \\ \text{because } I \text{ is of size } n \times n}}{=} (-5)^n \det(I) = (-5)^n$

Question 10a. *Write a full calculation/argumentation clearly in the box.*

We have

$$T_1(\vec{0})=\vec{0} \text{ and } T_2(\vec{0})=\vec{0}$$

Therefore $T_1(\vec{0})=T_2(\vec{0})$

Question 10b. *Write a full calculation/argumentation clearly in the box.*

We need to verify the three properties of the definition of a subspace:

(1) By part (a) $\vec{0} \in H$

(2) Let $\vec{u}, \vec{v} \in H$ we need to show that $\vec{u}+\vec{v} \in H$.

Since $\vec{u} \in H$ we have: $T_1(\vec{u})=T_2(\vec{u})$

Since $\vec{v} \in H$ we have: $T_1(\vec{v})=T_2(\vec{v})$

Thus

$$T_1(\vec{u}+\vec{v})=T_1(\vec{u})+T_1(\vec{v})=T_2(\vec{u})+T_2(\vec{v})=$$
$$=T_2(\vec{u}+\vec{v})$$

$\Rightarrow \vec{u}+\vec{v} \in H$

(3) Let $\vec{u} \in H$ and $c \in \mathbb{R}$ we need to show that $c\vec{u} \in H$.

Since $\vec{u} \in H$ we have $T_1(\vec{u})=T_2(\vec{u})$

Thus

$$T_1(c\vec{u})=cT_1(\vec{u})=cT_2(\vec{u})=T_2(c\vec{u})$$

$\Rightarrow c\vec{u} \in H$

This proves that H is a subspace of $\mathbb{R}^n$