UNIVERSITY OF TWENTE.

Course	: Linear Algebra	Date	: 4 April 2025
Module	: 2A	Time	: 13:45 – 15:45
Course code	: 202001205	Reference	: BIT, BMT, CE, CSE, IDE & IEM

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Linear Algebra

Exam

Instructions

This exam contains 10 questions. You shall use the attached *answer form* to submit your answers.

- For questions 1–4, you are only required to fill in the **final answer** on the answer form.
- For questions 5–10, you are required to write down a **full calculation and argumentation**.

You will hand in your answer form only. **Any text outside the answer form will not be considered.**
If you run out of space, you can use the extra space at the end of the answer form. Refer clearly to that space in the original answer.
Do not write with red pen or pencil. Do not use correction fluid or tape.
The use of electronic devices is not allowed!

Final answer questions

Write only your final answer on the answer form.

1. For each of the following statements, determine whether it is true or false. [3 pt]

(a) Every consistent linear system that has a free variable must have infinitely many solutions.

(b) For any three linearly dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$ we have that $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, for some $c_2, c_3 \in \mathbb{R}$.

(c) Every matrix of size 2×2 has a real eigenvalue.

(d) For any two square matrices A and B of the same size we have that $\det(A^TB) = \det(AB^T)$.

2. Find all $a, b, c \in \mathbb{R}$ such that the polynomial $f(x) = ax^2 + bx + c$ satisfies $f(2) = -8$, $f(-1) = 1$, and $f(0) = -2$. [3 pt]

3. A matrix A and its row reduced echelon form R are given below.

$$A = \begin{pmatrix} 4 & 4 & 1 & 9 & 1 \\ 2 & 2 & 1 & 5 & 0 \\ 3 & 3 & 1 & 7 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Determine the parametric vector form of the solution set of the system $A\mathbf{x} = \mathbf{0}$. [2 pt]

(b) Find a basis for $\mathrm{Null}\, A$. [2 pt]

(c) Determine the dimension of $\mathrm{Null}\, A$. [1 pt]

4. Let A and B be invertible matrices of size 3×3 such that $AA^T = I$ and $B^{-1} + B^T = 3I$. Solve for X the matrix equation [2 pt]

$$(BX^{-1}A)^{-1} + (BX^TA)^T = 5I.$$

Continues on the following page.

Open questions

The full solutions to questions 5–10 must be clearly written down on the answer form, including calculations and argumentations.

Points will not be awarded for reaching a correct result if this is not supported by a correct procedure and by a sound and clear argumentation.

5. Find $x_1, x_2, x_3 \in \mathbb{R}$ such that the matrix $A = \begin{pmatrix} 2 & x_1 \\ x_2 & x_3 \end{pmatrix}$ satisfies both properties below: [4 pt]

$$\operatorname{Null} A = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \quad \text{and} \quad \operatorname{Col} A = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}.$$

6. Compute the determinant of the following matrix: $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$. [3 pt]

7. Consider the basis S of $\mathbb{R}^3$ given by $S = \left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 6 \end{pmatrix}\right\}$.

(a) Compute the coordinates $[\mathbf{v}]_S$ of the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with respect to the basis S. [2 pt]

(b) Let A be a matrix of size 3×3 that has eigenvalues $\frac{1}{2}$ and $\frac{1}{3}$ with corresponding eigenspaces [2 pt]

$$E_{\frac{1}{2}} = \operatorname{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right\} \quad \text{and} \quad E_{\frac{1}{3}} = \operatorname{Span}\left\{\begin{pmatrix} -3 \\ 1 \\ 6 \end{pmatrix}\right\}.$$

Compute the vector $A\mathbf{w}$, where the coordinates of $\mathbf{w}$ with respect to S are $[\mathbf{w}]_S = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

(Hint: You do not need to explicitly compute A; the vectors in S are eigenvectors of A).

8. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{e}_1$ and $T_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{e}_2$, and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the *clockwise* rotation by $\frac{\pi}{4}$ radians around the origin.

(a) Determine the representation matrix of T_1^{-1}. [1 pt]

(b) Determine the representation matrix of the linear transformation $T = T_2^{-1} \circ T_1^{-1}$. [2 pt]

(c) Explain why T is onto. [1 pt]

9. Let A be a matrix of size $n \times n$ such that $A(A + 5I) = 0$.

(a) Show that for all $\mathbf{v} \in \mathbb{R}^n$ we have that $(A + 5I)\mathbf{v}$ belongs to $\operatorname{Null} A$. [2 pt]

(b) Prove that if A is invertible, then $\det A = (-5)^n$. [2 pt]

10. Fix some $n > 0$. Let $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^n$ be two linear transformations.

(a) Provide a brief explanation of why we must have that $T_1(\mathbf{0}) = T_2(\mathbf{0})$. [1 pt]

(b) Prove that the set $H = \{\mathbf{x} \in \mathbb{R}^n \mid T_1(\mathbf{x}) = T_2(\mathbf{x})\}$ is a subspace of $\mathbb{R}^n$. [3 pt]

Total: 36 pt

1	2	3	4	5	6	7	8	9	10	Σ

Do not fill in this table.

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Only for sorting.

Answer form

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Full name in BLOCK LETTERS _____

Student number _____

Programme _____

⚠ **FILL IN YOUR DATA AS SOON AS YOU RECEIVE THE EXAM** ⚠

Question 1. *Check the correct answers.*
(four correct answers = 3pt, three correct answers = 2pt, two correct answers = 1pt, fewer than two correct answers = 0pt)

(a) ☐ True ☐ False
(b) ☐ True ☐ False
(c) ☐ True ☐ False
(d) ☐ True ☐ False

Question 2.

a = b = c =

Question 3.

(a) The parametric vector form is

(b) A basis for $\mathrm{Null}\, A$ is

(c) The dimension of $\mathrm{Null}\, A$ is

Question 4.

X =

Question 5. *Write a full calculation/argumentation clearly in the box.*

Question 6. *Write a full calculation/argumentation clearly in the box.*

Question 7a. *Write a full calculation/argumentation clearly in the box.*

Question 7b. *Write a full calculation/argumentation clearly in the box.*

Question 8a. *Write a full calculation/argumentation clearly in the box.*

Question 8b. *Write a full calculation/argumentation clearly in the box.*

Question 8c. *Write a full calculation/argumentation clearly in the box.*

Question 9a. *Write a full calculation/argumentation clearly in the box.*

Question 9b. *Write a full calculation/argumentation clearly in the box.*

Question 10a. *Write a full calculation/argumentation clearly in the box.*

Question 10b. *Write a full calculation/argumentation clearly in the box.*

Additional writing space. *Clearly refer to this space in the original answer*

Additional writing space. *Clearly refer to this space in the original answer*

Additional writing space. *Clearly refer to this space in the original answer*