

Do not fill in!

Exercise	1	2	3	4	5	6	Σ
Points							

Name:

Student number:

Course : **Discrete Mathematics for Computer Science; Exam**

Date : November 4th, 2025

Time : 8:45 – 10:45

Motivate all your answers and give proofs if applicable, unless otherwise stated.

Write your solutions in the designated boxes.

The use of electronic devices is not allowed.

1. [3+2+1 pt]

(a) Let, the following symbols represent these primitive statements

Symbol	Statement
p	I have to work
q	it rains
r	I go to the amusement park
s	I go to the football match
t	I go to the museum

Determine for each of the statements in symbolic form if it is equivalent to the following natural language statement.

If I don't have to work and it doesn't rain, I will go to the amusement park or to the football match. If I don't have to work and it does rain, I will go to the museum.

Statement	Equivalent
$[(\neg p \wedge \neg q) \rightarrow (r \vee s)] \wedge [(\neg p \wedge q) \rightarrow t]$	yes / no
$[(\neg p \wedge \neg q) \rightarrow (r \vee s)] \vee [(\neg p \wedge q) \rightarrow t]$	yes / no
$\neg p \rightarrow [(\neg q \rightarrow (r \vee s)) \wedge (q \rightarrow t)]$	yes / no
$\neg p \rightarrow [(\neg q \rightarrow (r \vee s)) \vee (q \rightarrow t)]$	yes / no
$\neg p \rightarrow [r \vee s \vee t]$	yes / no

You do not need to motivate your answer.

Grading :
-1.5 pts per
wrong/missing
answer.
No less than
0 points.

(b) Prove the following equivalence or give a counterexample.

$$(p \wedge q) \vee \neg(\neg p \vee q) \iff p$$

For a proof, use the **laws of logic**. Use only one law for each step.

$(p \wedge q) \vee \neg(\neg p \vee q)$	
$\iff (p \wedge q) \vee (\neg\neg p \wedge \neg q)$	De Morgan
$\iff (p \wedge q) \vee (p \wedge \neg q)$	Double negation
$\iff p \wedge (q \vee \neg q)$	Distributivity
$\iff p \wedge T$	Inverse
$\iff p$	Identity

(c) Prove the following equivalence or give a counterexample.

$$(p \wedge q) \vee (\neg r \wedge s) \iff (\neg r \vee q) \wedge (p \vee s)$$

For a proof, use the **laws of logic**. Use only one law for each step.

The equivalence does not hold.	
Counter-examples are $p=1, q=s=t=0$	
or $p=0, q=s=t=1$.	
For the first, we obtain	
$(1 \wedge 0) \vee (1 \wedge 0)$	$\iff (1 \vee 0) \wedge (1 \vee 0)$
$0 \vee 0$	$\iff 1 \wedge 1$
0	$\iff 1$
Grading: $\frac{1}{2}$ pt for a correct counter-example.	
$\frac{1}{2}$ pt for verifying it.	

2. [3+3 pt]

(a) Prove the following statement

$$[(p \vee q) \wedge (\neg p \vee q) \wedge (q \rightarrow r) \wedge \neg(q \wedge p) \wedge \neg s] \rightarrow [(p \wedge r) \rightarrow s]$$

Use the **rules of inference** and/or the **laws of logic**. Use only one rule or law for each step and explain which previous steps you used.

Step	Steps used
(1) $(\neg p \vee q) \wedge \neg(q \wedge p)$	premise
(2) $(\neg p \vee q) \wedge (\neg p \vee \neg q)$	(1)
(3) $\neg p \vee (q \wedge \neg q)$	(2)
(4) $\neg p \vee F_0$	(3)
(5) $\neg p$	(4)
(6) $\neg p \vee \neg r$	(5)
(7) $\neg p \vee \neg r \vee s$	(6)
(8) $\neg(p \wedge r) \vee s$	(7)
(9) $\therefore (p \wedge r) \rightarrow s$	(8)
<p>Grading comment: Intermediate steps do <u>not</u> give partial points. Invented premises, clear short cuts (i.e., writing the conclusion after getting stuck), and inference of incorrect statements to short cut to the conclusion result in 0 points.</p>	

(b) Verify the validity of the following argument.

$$\begin{array}{l}
 \forall x [r(x) \vee s(x)] \\
 \exists x [r(x) \leftrightarrow s(x)] \\
 \forall x [r(x) \rightarrow p(x)] \\
 \forall x [s(x) \rightarrow q(x)] \\
 \hline
 \therefore \exists x [p(x) \wedge q(x)]
 \end{array}$$

For verification use the **rules of inference** and/or the **laws of logic**. Use only one rule or law for each step and explain which previous steps you used.

Step	Steps used
(1) $\exists x [r(x) \leftrightarrow s(x)]$	premise
(2) $\forall x [r(x) \vee s(x)]$	premise
(3) $\forall x [r(x) \rightarrow p(x)]$	premise
(4) $\forall x [s(x) \rightarrow q(x)]$	premise
(5) $r(a) \rightarrow s(a)$ for some a	(1)
(6) $s(a) \rightarrow r(a)$ for the same a in (5)	(1)
(7) $r(a) \vee s(a)$	(2)
(8) $s(a)$	(5), (7)
(9) $r(a)$	(8), (6)
(10) $r(a) \rightarrow p(a)$	(3)
(11) $p(a)$	(9), (10)
(12) $s(a) \rightarrow q(a)$	(4)
(13) $q(a)$	(8), (12)
(14) $p(a) \wedge q(a)$	(11), (13)
(15) $\therefore \exists x [p(x) \wedge q(x)]$	(14)
Grading comment if the quantifiers are not clearly addressed (e.g., writing only quantified statements or writing no quantified statements): Opts.	

3. [3+3 pt]

(a) For each $i \in \mathbb{Z}^+$, let $A_i = \{x \in \mathbb{R} \mid (x^2 < i) \wedge (x > -1 - \frac{1}{i})\}$.

Determine $\bigcap_{i=3}^{\infty} A_i$.

$$\bigcap_{i=3}^{\infty} A_i = [-1, \sqrt{3}).$$

Possible proof: $x \geq \sqrt{3} \Rightarrow x^2 \geq 3 \Rightarrow x \notin A_3$.

$x < -1 \Rightarrow \exists i \in \mathbb{Z} : i \geq \frac{1}{-1-x} \Rightarrow -1-x \geq \frac{1}{i}$

Then $x \leq -1 - \frac{1}{i}$, so $x \notin A_i$.

Let $x \in [1, \sqrt{3})$ and $i \in \mathbb{Z}, i \geq 3$. Then $x^2 < 3 \leq i$.

Moreover, $x \geq -1 \geq -1 - \frac{1}{i}$. Hence $x \in A_i$.

We conclude that $x \in \bigcap_{i=3}^{\infty} A_i$.

(b) Consider a universe $\mathcal{U}$ and let $A, B, C \subseteq \mathcal{U}$. Either prove using *proof by contradiction*, or disprove the following claim:

$$\text{If } A - (B \cup \bar{C}) \subseteq B, \text{ then } A \cap C \subseteq B.$$

A direct proof
gives (at most)
1 pt.

We assume $A \cap C \not\subseteq B$ and $A - (B \cup \bar{C}) \subseteq B$.

The former means $\exists x \in A \cap C$ with $x \notin B$.

↑ important - it does not hold for all $x \in A \cap C$.

Hence, $x \in A$, $x \in C$, $x \notin B$. Thus, $x \notin \bar{C}$
and so $x \notin B \cup \bar{C}$.

This implies $x \in A - (B \cup \bar{C})$ and the
second assumption now gives $x \in B$,

a contradiction (to $x \notin B$ observed before).

4. [3+3 pt]

- (a) Identify the smallest possible $n_0 \in \mathbb{Z}^+$, such that for all $n \in \mathbb{Z}^+$ with $n \geq n_0$ the following holds:

$$2^n \geq n^2.$$

Prove that n_0 is indeed the smallest integer such that the above holds.

$2^n \geq n^2$ holds for $n=1, 2$ and 4 but not for $n=3$ ($2^3=8 < 9=3^2$).

We prove by mathematical induction that $2^n \geq n^2$ holds for all $n \geq 4$:

- Base Case: For $n=4$, $2^4=16 \geq 16=4^2$.

- Induction Hypothesis: Assume that for a fixed, arbitrary $k \in \mathbb{Z}^+$, $k \geq 4$, $2^k \geq k^2$ holds.

- Induction step: Consider $k+1$

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \stackrel{\text{IH}}{\geq} 2 \cdot k^2 = k^2 + k^2 \\ &\geq k^2 + 4k = k^2 + 2k + 2k \geq k^2 + 2k + 1 \\ &\quad \uparrow \quad \quad \quad \uparrow \\ &\quad \text{because} \quad \quad \quad \text{because} \\ &\quad k \geq 4 \quad \quad \quad k \geq 4 \end{aligned}$$

$= (k+1)^2$. It follows that $2^n \geq n^2$ for all $n \in \mathbb{Z}^+$ with $n \geq 4$.

Comment: An induction step from $k=2$ to $k=3$ cannot work (since $2^3 < 3^2$).

Comment: It is important to argue for all possible balanced strings of length $2(k+1)$, not just the ones that can be built by applying Rule ii or iii to a string of length $2k$.

(b) We say a binary string is *balanced* if and only if it has positive even length, and it can be built using the following rules:

- The string "01" is the only balanced binary string of length 2.
- If S is balanced then so is "0" + S + "1" (the concatenation of "0", string S and "1").
- If S and T are balanced then so is $S+T$ (the concatenation of balanced strings S and T).

Use strong induction to show that for every $n \in \mathbb{Z}^+$, every balanced string of length $2n$ has an equal number of 0's and 1's.

You may use $|S|$ to denote the length of string S .

We use strong induction over n :

- Base Case: For $n=1$, it is already given that "01" is the only balanced binary string of length 2, and it has an equal number of "0" and "1".

- Induction Hypothesis: Assume that, for a fixed but arbitrary $k \geq 1$, all balanced strings of length between 2 and $2k$ have an equal number of "0" and "1".

- Induction Step: Consider an arbitrary balanced string Q of length $2(k+1) \geq 2$. By definition of balanced strings, there are two cases how Q can be built:

- either "0" + S + "1", where S is a balanced string with $|S|=2k$. In this case S has k "0"s and k "1"s by the IH, and therefore Q has $k+1$ "0"s and $k+1$ "1"s.

- or $S+T$, where S and T are balanced strings with respective lengths $2u, 2l$ with $1 \leq u, l \leq k$. By IH, S has u "0"s and "1" each and T has l "0"s and "1"s each. Thus $Q=S+T$ has $u+l$ "0"s and $u+l$ "1"s.

5. [2+2+2 pt] Recall that for a function $h : A \rightarrow B$, A is its domain and B is its co-domain.

- (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. Prove that f is not one-to-one. Redefine its domain so that it becomes one-to-one, and prove that this is the case.

$f(1) = f(-1) = 1$, so f is not one-to-one.

Redefinition (one possibility): $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$.

Proof that it is one-to-one: let $f(x) = f(y)$.

Then $x^2 = y^2$, so $x = \pm y$. But $x, y \geq 0$,
so $x = y$.

[Alternatively: argue that f is strictly increasing
on $(0, \infty)$]

- (b) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(x) = e^x$. Prove that g is not onto. Redefine its co-domain so that it becomes onto, and prove that this is the case.

$g(x) = e^x = 0$ has no solution for x , so
 g is not onto.

Redefinition: $g : \mathbb{R} \rightarrow \mathbb{R}^+$.

Proof that it is onto: let $y \in \mathbb{R}^+$. Then
 $g(\ln y) = e^{\ln y} = y$, where $\ln y \in \mathbb{R}$.

- (c) Consider a finite set B . For all $i \in \mathbb{Z}$, with $1 \leq i \leq 2^{|B|} + 1$, let $A_i \subseteq B$ be a subset of B . Prove that there must exist indices k, ℓ , with $1 \leq k, \ell \leq 2^{|B|} + 1$ and $k \neq \ell$, such that $A_k = A_\ell$.

B has $2^{|B|}$ distinct subsets.

By the pigeonhole principle, given $2^{|B|} + 1$ subsets of B , at least two of them must be equal.

6. [3+3 pt]

- (a) Let R_1 be a relation on A that is symmetric and transitive and furthermore satisfies the property that for every $a \in A$ there exists a $b \in A$ such that $(a, b) \in R_1$. Is R_1 an equivalence relation? Prove it or provide a counterexample.

We show that R_1 is reflexive:
It is given, that for every $a \in A$ there exists a $b \in A$ such that $(a, b) \in R_1$. Because R_1 is symmetric, also $(b, a) \in R_1$. By transitivity, since $(a, b) \in R_1$ and $(b, a) \in R_1$, we also have $(a, a) \in R_1$. Therefore R_1 is reflexive. Since it is also symmetric and transitive, it is an equivalence relation.

- (b) Consider the relation R_2 on $\mathbb{Z}^+$, defined by

$$R_2 = \{(a, b) \mid a \text{ and } b \text{ have the same number of digits (in decimal)}\}.$$

Show that R_2 is an equivalence relation and give all corresponding equivalence classes. We count the digits of integers without leading zeroes.

R_2 is:
- reflexive: every element of $\mathbb{Z}^+$ has the same number of digits as itself
- symmetric: if $a \in \mathbb{Z}^+$ has the same number of digits as $b \in \mathbb{Z}^+$, then b also has the same number of digits as a . Thus $(a, b) \in R_2$ implies $(b, a) \in R_2$
- transitive: if $a \in \mathbb{Z}^+$ has the same number of digits as $b \in \mathbb{Z}^+$ and b has the same number of digits as $c \in \mathbb{Z}^+$ then a and c also have the same number of digits.

Total: 36 points

6b continued) Therefore R_2 is an equivalence relation.

Each equivalence class is comprised of the elements of $\mathbb{Z}^+$ with the same number of digits.

That is, $\forall i \in \mathbb{Z}$, we have class

$$[10^i] = \{x \in \mathbb{Z}^+ \mid x \text{ has } i+1 \text{ many digits}\}.$$