Course : **Discrete Mathematics for Computer Science; Exam**
Date : Tuesday, October 29th, 2024
Time : 8:45–10:45

Motivate all your answers and give proofs if applicable, unless otherwise stated.
Write your solutions in the designated boxes.
The use of electronic devices is not allowed.

1. [1+2+3pt] *You do not need to motivate your answer for parts* (a) *and* (b) *of this question.*

   (a) Let the universe comprise all students at the UT. We define the following open statements

   * $d(x)$ student x is enrolled in the course Discrete Mathematics
   * $s(x)$ student x studied the course material and did the exercises for Discrete Mathematics.
   * $e(x)$ student x passes the exam in Discrete Mathematics

   Express the following statement in symbolic form:

   All enrolled students of the course Discrete Mathematics pass the exam if they studied the course material and did the exercises.

Do not fill in!

Exercise	1	2	3	4	5	6	$\sum$
Points							

(b) Let the universe comprise all guests to a party. We define the following open statements

* $b(x)$ guest x came to the party by bike
* $c(x)$ guest x came to the party by car

For this part you may use $(x \neq y)$ for the open statement that is true if and only if x and y are different elements of the universe. Express the following statement in symbolic form:

All guests to the party came by bike except one, who came by car.

(c) Let the universe comprise all positive integers. Negate the following statement and transform it into a statement without a negation.

$\forall x\,[(x \geq 12) \rightarrow \exists a \exists b (x = 3a + 7b)]$.

Use only one rule or law for each step of your transformation.

Step

2. [3+3pt]

(a) Verify the validity of the following argument using laws of logic and rules of inference. Use only one rule or law for each step and explain which previous steps you used.
$[p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)] \rightarrow (s \vee t)$

Step	Explanation

(b) Let $p(x)$, $q(x)$, $r(x)$, and $s(x)$ be arbitrary open statemts over variables of an arbitrary universe. Verify the validity of the following argument using laws of logic and rules of inference.
Use only one rule or law for each step and explain which previous steps you used. Give an explanation if you introduce or use new variables or quantifiers.

$$\begin{array}{l} \forall x\,[p(x) \to (q(x) \wedge r(x))] \\ \underline{\forall x\,[p(x) \wedge s(x)]} \\ \therefore \forall x\,[s(x) \wedge r(x)] \end{array}$$

Step	Explanation

3. [2+4pt]

(a) Let $A, B, C \subseteq \mathcal{U}$. Prove or disprove the following:

$$A - C = B - C \Rightarrow A = B$$

(b) Let $A, B \subseteq \mathcal{U}$. Prove or disprove the following:

$$A \subseteq B \Leftrightarrow A \cap B = A$$

4. [3+3pt]

(a) For all $n \in \mathbb{Z}^+$ with $n \geq 12$, show that n can be written as a sum of 3's and/or 7's.

(b) Prove that for any integer $n \geq 2$,

$$1 + 4 + 7 + 10 + 13 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}.$$

5. [2+2+1+1pt]

Let $f : \mathbb{Z} \to \mathbb{Z}^+$ be defined by

$$f(x) = \begin{cases} 2x + 1, \text{ if } x \geq 0 \\ -2x \text{ otherwise.} \end{cases}$$

(a) Prove or disprove: f is one-to-one.

(b) Prove or disprove: f is onto.

(c) Prove or disprove: f is invertible.

(d) Let A, B be two finite sets with $|A| > |B|$. Use the pigeonhole principle in order to prove that no function $g : A \to B$ can be one-to-one.

6. [2+2+2pt]

(a) Let $R_1 \subseteq A \times A$, with $A = \{-2, 1, 2, 3, 5\}$ be given by

$$R_1 = \{(-2,-2), (-2,1), (-2,2), (1,-2), (1,1), (1,2), (1,5), (2,-2), \\ (2,1), (2,2), (2,5), (3,5), (5,1), (5,2), (5,3), (5,5)\}.$$

Is R_1 transitive, symmetric, antisymmetric, and/or reflexive? Justify your answers.

(b) Let W be a set with $|W| \geq 2$ and let R_2 be the binary relation on $\mathcal{P}(W)$ defined by:

$$(A, B) \in R_2 \Leftrightarrow A \cap B \neq \varnothing.$$

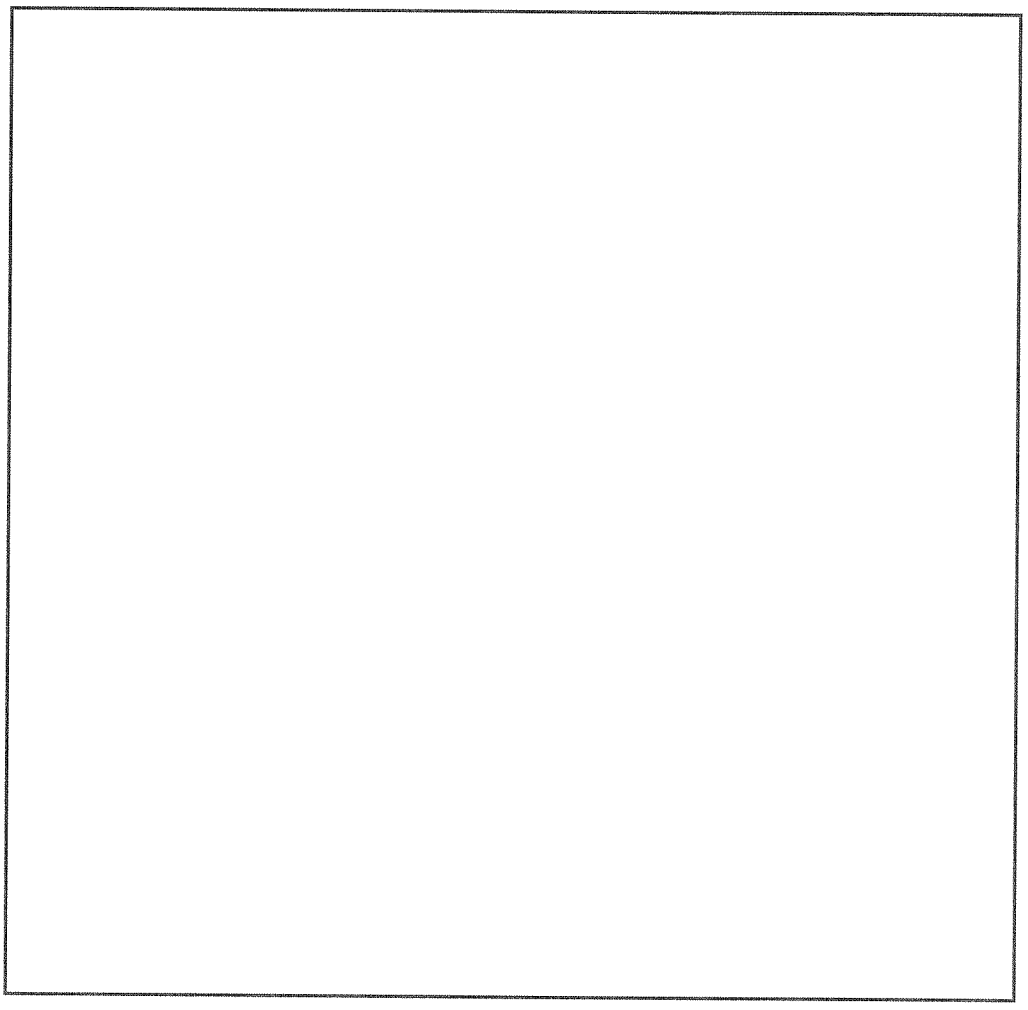
Is R_2 transitive, symmetric, antisymmetric, and/or reflexive? Justify your answers.

(c) Let R_3 be the (partial order) relation on $\mathbb{Z}^+$ defined by

$$R_3 = \{(a, b) \mid a \text{ divides } b\}.$$

Draw the Hasse diagram representing R_3, on the set

$$A = \{1, 2, 3, 4, 6, 8, 12, 24\}.$$



Total: 36 points