Course : **Mathematics B2 (Newton)**

Date : January 13, 2016
Time : 13.45 - 15.45

Motivate all answers and calculations.
The use of electronic devices is not permitted.

[3p] **1.** a) Show by computation that for positive $a \in \mathbb{R}$

$$\lim_{x\to a} \frac{x-a}{\sqrt{x-a+3}-\sqrt{3}} = 2\sqrt{3}$$

Use l'Hopital's rule [1/2 p]

and check '0/0' [1/2 p]

Then

$$\lim_{x\to a} \frac{1}{\frac{1}{2\sqrt{x-a+3}}} \qquad [3/2p]$$

$$= 2\sqrt{3} \qquad [1/2p]$$

Alternatively, 'square root trick'

$$\lim_{x\to a} \frac{x-a}{\sqrt{x-a+3}-\sqrt{3}} \cdot \frac{\sqrt{x-a+3}+\sqrt{3}}{\sqrt{x-a+3}+\sqrt{3}} \qquad [1p]$$

$$= \lim_{x\to a} \frac{(x-a)(\sqrt{x-a+3}+\sqrt{3})}{x-a} \qquad [1p]$$

$$= 2\sqrt{3} \qquad [1p]$$

[2p] b) For which real value p is de function

$$f(x) = \begin{cases} px & \text{als} \quad x \le a \\ \dfrac{x-a}{\sqrt{x-a+3}-\sqrt{3}} & \text{als} \quad x > a \end{cases}$$

continuous in every x?

Remark that $f(x)$ is continuous at every $x \neq a$ [1/2 p]

Continuity at $x = a$ requires

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a) \qquad [1/2p]$$

Therefore

$$pa = 2\sqrt{3} = pa \qquad [1/2p]$$

Hence

$$p = \frac{2\sqrt{3}}{a} \qquad [1/2p]$$

The case $a = 0$ is not part of the exercise.

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2. The function f is given by $f(x) = \sin(\cos(x))$.

[2p] a) Determine $f'(x)$.

Note

$$\frac{d}{dx}(\sin(x)) = \cos(x) \quad \text{and} \quad \frac{d}{dx}(\cos(x)) = -\sin(x) \qquad [1/2p]$$

With the chain rule

$$\frac{d}{dx}(\sin(\cos(x)) = \cos(\cos(x)) \cdot (-\sin(x)) \qquad [1p]$$

yielding

$$f'(x) = -\sin(x)\cos(\cos(x)) \qquad [1/2p]$$

Alternatively; answering directly and correctly using the chain rule gives full score as well

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[2p] b) Determine the linearisation of $f(x)$ in $x = \pi/4$.

We note that

$$f(\pi/4) = \sin(\frac{1}{2}\sqrt{2}) \quad \text{and} \quad f'(\pi/4) = -\frac{1}{2}\sqrt{2}\cos(\frac{1}{2}\sqrt{2}) \qquad [1p]$$

The linearisation $L(x)$ may be written as

$$L(x) = \sin(\frac{1}{2}\sqrt{2}) - \frac{1}{2}\sqrt{2}\cos(\frac{1}{2}\sqrt{2}) \cdot (x - \frac{\pi}{4}) \qquad [1p]$$

[4p] **3.** Determine all extreme values (global and local) of the function $f(x) = xe^{-2x}$ on the interval $(0, 4]$.

Candidate extreme values are located at critical points

$$f'(x) = 0 \quad \leftrightarrow \quad e^{-2x} - 2xe^{-2x} = 0 \quad \leftrightarrow \quad x = \frac{1}{2} \qquad [1p]$$

We observe $f'(x) > 0$ for $x < 1/2$ and $f'(x) < 0$ for $x > 1/2$. Hence, f has a global maximum at $x = 1/2$ given by $f(1/2) = 1/(2e)$ [1p]

f has a minimum at boundary $x = 4$ given by $f(4) = 4e^{-8}$ [1p]

This minimum is local, not global, since

$$\lim_{x\to 0^+} f(x) = 0 \qquad [1p]$$

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4. Given

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x^2+y^4} & \text{als} \quad (x,y) \neq (0,0) \\ 0 & \text{als} \quad (x,y) = (0,0) \end{cases}$$

[2p] a) Is f continuous in $(0,0)$?

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f is continuous at $(0,0)$ only in case

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \qquad [1/2p]$$

But this limit does not exist; for example, if (x,y) is tending to $(0,0)$ along the y-axis we have [1/2 p]

$$\lim_{y\to 0} \frac{0+y^2}{0+y^4} = \lim_{y\to 0} \frac{1}{y^2} = \infty \qquad [1/2p]$$

Since this limit clearly is not 0, f is not continuous at $(0,0)$ [1/2 p]

[3p] b) Determine the equation for the tangent plane to the graph of $f(x,y)$ at the point $(2,1,1)$.

We compute

$$\frac{\partial}{\partial x}\Big(f(x,y)\Big) = \frac{2x(y^4-y^2)}{(x^2+y^4)^2}; \quad \frac{\partial}{\partial x}\Big(f(2,1)\Big) = 0 \quad [1p]$$

and

$$\frac{\partial}{\partial y}\Big(f(x,y)\Big) = \frac{2yx^2 - 2y^5 - 4y^3x^2}{(x^2+y^4)^2}; \quad \frac{\partial}{\partial y}\Big(f(2,1)\Big) = -\frac{2}{5} \qquad [1p]$$

Hence, the equation for the tangent plane is

$$z = f(2,1) + 0 \cdot (x-2) - \frac{2}{5} \cdot (y-1) \qquad [1/2p]$$

i.e., with $f(2,1) = 1$ we find

$$z = \frac{7}{5} - \frac{2}{5}y \qquad [1/2p]$$

[3p] **5.** a) Given is the function $f(x) = x^3 - 2/x$ for $1 \le x \le 3$. We divide the interval $[1,3]$ in n equal sub-intervals. Give the expression for the Riemann sum of the function f in case we choose the right-most point of each sub-interval for evaluate f.

Riemann sum wtih step size $h = 2/n$

$$\sum_{k=1}^{n} f(\text{right-hand boundary of } k\text{-th subinterval}) \cdot \frac{2}{n} \qquad [1p]$$

With

$$\text{right-hand boundary of } k\text{-th subinterval} = x_k = 1 + k \cdot \frac{2}{n} \qquad [1p]$$

yields as expression

$$\sum_{k=1}^{n} \Big((1+\frac{2k}{n})^3 - \frac{2}{1+\frac{2k}{n}}\Big) \cdot \frac{2}{n} \qquad [1p]$$

There is no need to simplify this further.

[3p] **6.** Determine $\frac{dy}{dx}$ in case

$$y(x) = \int_x^{x^2} \cos(t^3)\,\mathrm{d}t$$

Splitting up the integral

$$y(x) = \int_0^{x^2} \cos(t^3)dt - \int_0^{x} \cos(t^3)dt \qquad [1p]$$

we find

$$\begin{aligned} \frac{dy}{dx} &= \cos((x^2)^3) \cdot 2x - \cos(x^3) \cdot 1 && [3/2p] \\ &= 2x\cos(x^6) - \cos(x^3) && [1/2p] \end{aligned}$$

Also full scores if dy/dx is written down directly.

[2p] **7.** a) Compute

$$\int x^2 \ln(2x)\ dx$$

Integration by parts:

$$\begin{aligned} \int x^2 \ln(2x)dx &= \int \ln(2x) d(\frac{1}{3}x^3) && [1/2p] \\ &= \frac{1}{3}x^3 \ln(2x) - \int \frac{1}{3}x^3 d\ln(2x) && [1/2p] \\ &= \frac{1}{3}x^3 \ln(2x) - \int \frac{1}{3}x^2 dx && [1/2p] \\ &= \frac{1}{3}x^3 \ln(2x) - \frac{1}{9}x^3 + C && [1/2p] \end{aligned}$$

Alternatively, not explicitly using differentials

$$\begin{aligned} \int x^2 \ln(2x)dx &= \frac{1}{3}x^3 \ln(2x) - \int \frac{1}{3}x^3 \left(\frac{1}{x}\right) dx && [1p] \\ &= \frac{1}{3}x^3 \ln(2x) - \int \frac{1}{3}x^2 dx && [1/2p] \\ &= \frac{1}{3}x^3 \ln(2x) - \frac{1}{9}x^3 + C && [1/2p] \end{aligned}$$

[2p] b) Given is $\sinh(x) = (e^x - e^{-x})/2$. Compute

$$\int_{-1}^{1} \sinh(t)\,\mathrm{d}t$$

If a student recognizes that sinh is an odd function ($\sinh(-t) = -\sinh(t)$) and concludes that the integral is 0 then full score.

Altenative:

$$\int_{-1}^{1} \frac{e^t - e^{-t}}{2} dt = \left[\frac{e^t + e^{-t}}{2}\right]_{-1}^{1} \qquad [3/2p]$$

$$= \frac{e^1 + e^{-1}}{2} - \frac{e^{-1} + e^1}{2} = 0 \qquad [1/2p]$$

In the first line: [1 p] for a correct antiderivative and [1/2 p] for the correct integration boundaries.

[3p] c) Compute

$$\int_0^\infty \frac{e^{-x}}{1 + e^{-2x}}\ dx$$

Note

$$\int_0^\infty \frac{e^{-x}}{1 + e^{-2x}} dx = \lim_{a\to\infty} \int_0^a \frac{e^{-x}}{1 + e^{-2x}} dx \qquad [1/2p]$$

Finding an antidervative via substitution $e^{-x} = u$

$$\int \frac{e^{-x}}{1 + e^{-2x}} dx = \int \frac{-du}{1 + u^2} = -\tan^{-1}(u) = -\tan^{-1}(e^{-x}) \qquad [1p]$$

Hence,

$$\lim_{a\to\infty} \Big(-\tan^{-1}(e^{-a}) + \tan^{-1}(e^0)\Big) \qquad [1/2p]$$

$$= -\tan^{-1}(0) + \tan^{-1}(1) = 0 + \frac{\pi}{4} = \frac{\pi}{4} \qquad [1p]$$

[2p] **8.** a) Compute

$$\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^k$$

This is a geometric series with first term 8/3 and ratio 2/3 [1 p]

Hence

$$\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^k = \frac{\text{first term}}{1-\text{ratio}} = \frac{8/3}{1/3} = 8 \qquad [1p]$$

[3p] b) Determine the McLaurin series for $1/(1-2x)^2$ by differentiating the geometric series $\sum_{n=0}^{\infty}(2x)^n$.

$\sum_{n=0}^{\infty}(2x)^n$ converges if $|2x| < 1$, so $-1/2 < x < 1/2$ [1/2 p]

For these x we have

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + ... \qquad [1/2p]$$

Hence

$$\frac{d}{dx}: \quad \frac{2}{(1-2x)^2} = 0 + 2 + 2(2x)^1 \cdot 2 + ... + n(2x)^{n-1} \cdot 2 + ... \qquad [1p]$$

We conclude

$$\frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} n(2x)^{n-1} \qquad [1p]$$

Total: 36 points