Test Probability Theory for TCS-BIT, June 12, 2020.

You may consult the reader and your own notes. A conventional calculator is allowed, but a programmable calculator is not. Always provide your arguments. For your convenience, a formula sheet and the normal table is provided with this test.

Read the following carefully: By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

We ask you to copy the following statement to your first answering sheet and sign it with your name and student number: "*I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.*"

1. Given the following joint probability distribution, i.e. the probabilities $P(X = x \text{ and } Y = y)$ of two random variables X and Y:

$x \setminus y$	-1	0	1
-1	1/15	1/15	2/15
0	1/15	1/15	0
1	1/3	1/5	1/15

   a) Determine the probability distribution of Y, and further find $E(Y)$ and $\text{Var}(Y)$.
   b) Determine the covariance between X and Y.
   c) Determine the probability distribution of $Z = X + Y$.
   d) Determine the conditional distribution of X given $Y = 1$.

2. The continuous random variable X has density function f given by

$$f(x) = \begin{cases} cx^4 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   a) Show that $c = 5$.
   b) Calculate $E[X]$ and $\text{Var}[X]$.
   c) Determine the density function of the random variable $Y = \sqrt{X}$.

3. In a given country, it is expected (from historical records) that one in every four (25%) inhabitants experiences an allergy during spring time. This year, a random sample of 1000 inhabitants is selected, and the number Y of individuals from the sample that developed an allergy, is recorded.

   a) What is the distribution of Y, if the historical 25% proportion is valid? Describe your assumptions.

b) The government of the country wants to develop a criterion in order to tell whether this year's spring-allergy episode has been specially heavy. It is decided that the criterion should look as follows: "a spring-allergy episode is called *heavy* if $Y > c$." Your task is to compute or approximate the value of c such that $P(Y > c) = 5\%$.

4. Fifty students volunteer to participate in a computer programming contest. Ten of these students are in fact outstanding programmers. For the contest, a team of ten students (out of the fifty volunteers) is selected completely at random, in the spirit of fairness. What is the probability that the selected team contains two or more outstanding programmers?

5. A national football team is known to have reached the final stage of the world cup tournament 20% of the time.

   a) Determine the distribution of the number of tournaments that the team needs to play before reaching a final stage. What is its expected value?

   b) Experts say that it makes a difference, whether we are talking about a "usual" generation of football players (where the previous figures apply), or a "golden" generation. In the case of a golden generation, they estimate that the team would reach the final stage with 70% probability. But unfortunately golden generations are rare: only one of every five generations are golden.What is the probability that the current generation of players is "usual" if the team reaches this year's final stage of the tournament? Define all relevant events and their (conditional) probabilities.

6. It has been determined that the speed of a typical car on a highway is normally distributed, with mean 110 Km/h and variance 15 Km/h.

   a) What is the probability that a randomly selected car drives at a speed between 100 and 120 Km/h?

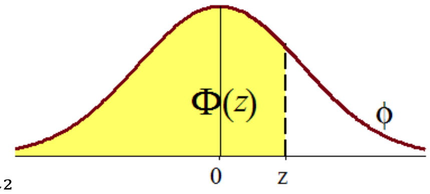
   b) If five cars are randomly and independently selected, what is the probability that at least three (i.e. 3, 4 or 5) have a speed between 100 and 120 Km/h?

Points:

Question	1a	1b	1c	1d	2a	2b	2c	3a	3b	4	5a	5b	6a	6b	Total
Points	4	2	2	2	2	2	4	2	4	4	2	4	2	4	40

Grade: $\frac{\text{Your Points}}{40} * 9 + 1$ (rounded to one decimal).

Standard normal probabilities



The tabel gives the distribution function Φ for a N(0, 1) variable Z

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx$$

The last column the N(0,1) gives the density function: $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

	Second decimal of *z*										
z	**0,00**	**0,01**	**0,02**	**0,03**	**0,04**	**0,05**	**0,06**	**0,07**	**0,08**	**0,09**	φ(*z*)
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359	0,3989
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753	0,3970
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141	0,3910
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517	0,3814
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879	0,3683
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224	0,3521
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549	0,3332
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852	0,3123
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133	0,2897
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389	0,2661
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621	0,2420
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830	0,2179
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015	0,1942
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177	0,1714
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319	0,1497
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441	0,1295
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545	0,1109
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633	0,0940
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706	0,0790
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767	0,0656
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817	0,0540
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857	0,0440
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890	0,0355
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916	0,0283
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936	0,0224
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952	0,0175
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964	0,0136
2,7	0,9965	0,9966	0,9967	0,9968	0,9969	0,9970	0,9971	0,9972	0,9973	0,9974	0,0104
2,8	0,9974	0,9975	0,9976	0,9977	0,9977	0,9978	0,9979	0,9979	0,9980	0,9981	0,0079
2,9	0,9981	0,9982	0,9982	0,9983	0,9984	0,9984	0,9985	0,9985	0,9986	0,9986	0,0060
3,0	0,9987	0,9987	0,9987	0,9988	0,9988	0,9989	0,9989	0,9989	0,9990	0,9990	0,0044
3,1	0,9990	0,9991	0,9991	0,9991	0,9992	0,9992	0,9992	0,9992	0,9993	0,9993	0,0033
3,2	0,9993	0,9993	0,9994	0,9994	0,9994	0,9994	0,9994	0,9995	0,9995	0,9995	0,0024
3,3	0,9995	0,9995	0,9995	0,9996	0,9996	0,9996	0,9996	0,9996	0,9996	0,9997	0,0017
3,4	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9998	0,0012
3,5	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,0009
3,6	0,9998	0,9998	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,0006

Formula sheet Probability Theory for BIT and TCS in module 4

Distribution	$\boldsymbol{E(X)}$	$\boldsymbol{var(X)}$
Geometric	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$n \cdot \frac{R}{N}$	$n \cdot \frac{R}{N} \cdot \frac{N-R}{N} \cdot \frac{N-n}{N-1}$
Poisson $P(X = x) = \frac{e^{-\mu}\mu^x}{x!}, x = 0, 1, 2, \ldots$	μ	μ
Uniform on (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang $f_X(x) = \frac{\lambda(\lambda x)^{n-1}e^{-\lambda x}}{(n-1)!}, x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$

$$var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} var(X_i) + \sum_{i \neq j}\sum cov(X_i, X_j)$$