Test Probability Theory for TCS-BIT, June 12, 2020.

You may consult the reader and your own notes. A conventional calculator is allowed, but a programmable calculator is not. Always provide your arguments. For your convenience, a formula sheet and the normal table is provided with this test.

Read the following carefully: By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

We ask you to copy the following statement to your first answering sheet and sign it with your name and student number:"*I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.*"

1. Political party "PP" initially supposes that one in every three voters (1/3) have a preference for their party. In this year's election, however, they obtained only 30% of the votes. This motivated the "PP" party to carry out a survey. For the survey, 1000 potential voters where asked for their preferences. Denote by X the number of survey participants who expressed a preference for the "PP" party.

   a) Give the distribution of the random variable X, under the party's initial assumption that 1/3 of voters have a preference for the "PP" party.

   b) Compute or approximate $P(X \leq 300)$ under the party's initial assumption.

2. Two (unbiased) dice are rolled. Denote by X the result observed in the first one, and by Y the result in the second one. Further, introduce $Z_1 = \min\{X, Y\}$ and $Z_2 = \max\{X, Y\}$.

   a) Prove that $P(Z_1 \geq i) = [P(X \geq i)]^2$. Use this to determine the distribution of Z_1, i.e. $P(Z_1 = i)$ for $i \in \{1, 2, 3, 4, 5, 6\}$.

   b) Give the joint distribution of Z_1 and Z_2 in a table. Are Z_1 and Z_2 independent?

   c) Find the conditional distribution of Z_2 given $Z_1 = 2$, i.e. $P(Z_2 = k|Z_1 = 2)$ for $k \in \{1, 2, 3, 4, 5, 6\}$.

   d) Denote by ρ the correlation between Z_1 and Z_2. Which of the following do you expect: $\rho < 0$, $\rho = 0$, or $\rho > 0$? Provide your argument without computing ρ.

3. The continuous random variable X has density function f given by

$$f(x) = \begin{cases} cx^2 & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

   a) Show that $c = 3/7$.

   b) Calculate $P[X \geq 1.5]$, $E[X]$ and $\text{Var}[X]$.

   c) Determine the distribution function $F(x) = P(X \leq x)$ and find the median M of X (the number M such that $P(X \leq M) = 1/2$).

d) Determine the density function of the random variable $Y = X^2$. Give first the range of Y, i.e. S_Y.

4. A Student must rush from home into the university. As he has no time, he chooses a wallet without knowing the amount and type of coins in it.

   a) Suppose that the wallet chosen by the student contains four types of coins: 10 cents, 20 cents, 50 cents and 1 Euro coins. Assume that the number of coins of a given type is a Poisson random variable with parameter $\mu = 5$, and suppose that these random variables are independent. Denoting by X the total number of coins in the wallet, determine $E[X]$ and $\text{Var}(X)$.

   b) In fact, the student had three wallets at home. If the student selects the first wallet, the situation is as described in (a). If he selects the second wallet, then it is again as in (a), but with the difference that μ (the parameter of the Poisson distribution) is equal to 2. For the third wallet, the situation is again like in (a) but with $\mu = 7$. If the student selects a wallet at random, what is the expected total number of coins in the wallet? Explain your assumptions.

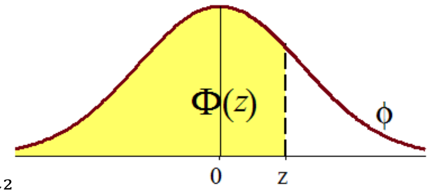
5. The weight (in Kilos) of a random adult person is assumed to be a continuous random variable which is uniformly distributed over the interval $[50, 110]$. The weights of different randomly chosen people are supposed to be independent of each other. One hundred (random) adults are waiting to board a plane. The plane is designed to work optimally for a total passenger-weight of 8500 Kilos. Compute or approximate the probability that the total weight of the one hundred adults is greater or equal than 8500.

Points:

Question	1a	1b	2a	2b	2c	2d	3a	3b	3c	3d	4a	4b	5	Total
Points	2	4	4	4	2	2	2	4	2	4	2	4	4	40

Grade: $\frac{\text{Your Points}}{40} * 9 + 1$ (rounded to one decimal).

Standard normal probabilities



The tabel gives the distribution function Φ for a N(0, 1) variable Z

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx$$

The last column the N(0,1) gives the density function: $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

	Second decimal of z										
z	**0,00**	**0,01**	**0,02**	**0,03**	**0,04**	**0,05**	**0,06**	**0,07**	**0,08**	**0,09**	φ(z)
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359	0,3989
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753	0,3970
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141	0,3910
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517	0,3814
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879	0,3683
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224	0,3521
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549	0,3332
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852	0,3123
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133	0,2897
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389	0,2661
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621	0,2420
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830	0,2179
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015	0,1942
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177	0,1714
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319	0,1497
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441	0,1295
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545	0,1109
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633	0,0940
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706	0,0790
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767	0,0656
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817	0,0540
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857	0,0440
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890	0,0355
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916	0,0283
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936	0,0224
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952	0,0175
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964	0,0136
2,7	0,9965	0,9966	0,9967	0,9968	0,9969	0,9970	0,9971	0,9972	0,9973	0,9974	0,0104
2,8	0,9974	0,9975	0,9976	0,9977	0,9977	0,9978	0,9979	0,9979	0,9980	0,9981	0,0079
2,9	0,9981	0,9982	0,9982	0,9983	0,9984	0,9984	0,9985	0,9985	0,9986	0,9986	0,0060
3,0	0,9987	0,9987	0,9987	0,9988	0,9988	0,9989	0,9989	0,9989	0,9990	0,9990	0,0044
3,1	0,9990	0,9991	0,9991	0,9991	0,9992	0,9992	0,9992	0,9992	0,9993	0,9993	0,0033
3,2	0,9993	0,9993	0,9994	0,9994	0,9994	0,9994	0,9994	0,9995	0,9995	0,9995	0,0024
3,3	0,9995	0,9995	0,9995	0,9996	0,9996	0,9996	0,9996	0,9996	0,9996	0,9997	0,0017
3,4	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9998	0,0012
3,5	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,0009
3,6	0,9998	0,9998	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,0006

Formula sheet Probability Theory for BIT and TCS in module 4

Distribution	$\boldsymbol{E(X)}$	$\boldsymbol{var(X)}$
Geometric	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$n \cdot \frac{R}{N}$	$n \cdot \frac{R}{N} \cdot \frac{N-R}{N} \cdot \frac{N-n}{N-1}$
Poisson $P(X = x) = \frac{e^{-\mu}\mu^x}{x!}, x = 0, 1, 2, \ldots$	μ	μ
Uniform on (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang $f_X(x) = \frac{\lambda(\lambda x)^{n-1}e^{-\lambda x}}{(n-1)!}, x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$

$$var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} var(X_i) + \sum_{i \neq j}\sum cov(X_i, X_j)$$