Test Probability Theory for TCS-BIT, June 12, 2020.

You may consult the reader and your own notes. A conventional calculator is allowed, but a programmable calculator is not. Always provide your arguments. For your convenience, a formula sheet and the normal table is provided with this test.

Read the following carefully: By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

We ask you to copy the following statement to your first answering sheet and sign it with your name and student number: "*I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.*"

1. Given the following joint probability distribution, i.e. the probabilities $P(X = x \text{ and } Y = y)$ of two random variables X and Y:

$x \setminus y$	0	1	2
0	1/9	1/9	1/9
1	0	1/9	2/9
2	2/9	1/9	0

   a) Determine the probability distribution of X, and further find $P(X \geq 1)$, $E(X)$, and $\text{Var}(X)$.

   b) Determine the correlation between X and Y.

   c) Are X and Y independent? Motivate your answer.

   d) Determine the conditional distribution of Y given $X = 1$ and calculate $E[Y|X = 1]$.

2. If X is $N(\mu, \sigma^2)$-distributed, then we say that $Z = e^X$ is log-normal(μ, σ^2)-distributed. These kind of random variables appear in financial and economic models.

a) Find the density function of Z (i.e. f_Z).

b) For the special case $\mu = 3$ and $\sigma^2 = 9$, find $P(1/2 \leq Z \leq 2)$.

c) Let X_1 and X_2 be independent and respectively $N(\mu_1, \sigma_1^2)$-distributed and $N(\mu_2, \sigma_2^2)$-distributed. Defining $Z_i = e^{X_i}$, for $i \in \{1, 2\}$, find the density of the product random variable $Z_1 \cdot Z_2$.

3. An empty train, consisting of three wagons, arrives at a station where eight people are waiting. These passengers choose at random, and independently of each other, which wagon to board.

   a) What is the probability that nobody boards in the first wagon?

   b) What is the probability that at most two people (i.e. 0, 1 or 2) end up in the second wagon?

c) Let X_1, X_2, X_3 denote respectively de number of people in the first, second, and third wagons. Are X_1, X_2, X_3 independent? Compute the probability of the event $A = \{X_1 = 2 \text{ and } X_2 = 4 \text{ and } X_3 = 2\}$.

4. A group of ten friends meets at the beach to play volleyball. Six of the ten friends are considered good players. In a usual beach volleyball match, two players play against two players. If these four players are chosen completely at random from the group of friends, what is the probability that the match will be fair, i.e. that both teams have the same number of good players?

5. Assume that the time (in years) between two consecutive iPhone releases is a random variable which is exponentially distributed with parameter $\lambda = 1$. These random variables are assumed to be independent of each other.

   a) Let S_n denote the total time until the n-th release of an iPhone (i.e. the iPhone n). Justify that the event "$S_n \leq 75$" is equal to the event "there are at least n iPhone releases in 75 years".
   b) Compute or approximate the probability of the event: "there are at least 60 iPhone releases in 75 years".

6. For the following questions, answer "True" or "False". If you answer "True", provide a proof, and if you answer "False", provide a simple counterexample.

   a) If two events A and B are independent, then they are mutually exclusive (i.e. disjoint).
   b) If $P(A|B) = P(A)$, then $P(B|A) = P(B)$.
   c) If X and Y are two random variables such that $E[XY] = 0$, then they are independent random variables.
   d) If X and Y are independent and positive random variables, then $E[X^Y] = E[X]^{E[Y]}$.
   e) If X is a random variable with $P[X \geq 2] > 0$, then $P[X \geq 3 | X \geq 2] \geq P[X \geq 3]$.

Points:

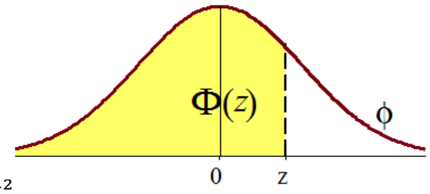
Question	1a	1b	1c	1d	2a	2b	2c	3a	3b	3c	4	5a	5b	6a	6b	6c	6d	6e	Total
Points	3	2	2	2	2	4	2	2	2	4	4	2	4	1	1	1	1	1	40

Grade: $\frac{\text{Your Points}}{40} * 9 + 1$ (rounded to one decimal).

Tab-6

Standard normal probabilities

The tabel gives the distribution function Φ for a N(0, 1) variable Z



$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx$$

The last column the N(0,1) gives the density function: $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

	Second decimal of *z*										
z	**0,00**	**0,01**	**0,02**	**0,03**	**0,04**	**0,05**	**0,06**	**0,07**	**0,08**	**0,09**	φ(z)
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359	0,3989
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753	0,3970
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141	0,3910
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517	0,3814
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879	0,3683
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224	0,3521
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549	0,3332
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852	0,3123
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133	0,2897
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389	0,2661
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621	0,2420
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830	0,2179
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015	0,1942
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177	0,1714
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319	0,1497
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441	0,1295
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545	0,1109
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633	0,0940
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706	0,0790
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767	0,0656
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817	0,0540
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857	0,0440
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890	0,0355
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916	0,0283
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936	0,0224
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952	0,0175
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964	0,0136
2,7	0,9965	0,9966	0,9967	0,9968	0,9969	0,9970	0,9971	0,9972	0,9973	0,9974	0,0104
2,8	0,9974	0,9975	0,9976	0,9977	0,9977	0,9978	0,9979	0,9979	0,9980	0,9981	0,0079
2,9	0,9981	0,9982	0,9982	0,9983	0,9984	0,9984	0,9985	0,9985	0,9986	0,9986	0,0060
3,0	0,9987	0,9987	0,9987	0,9988	0,9988	0,9989	0,9989	0,9989	0,9990	0,9990	0,0044
3,1	0,9990	0,9991	0,9991	0,9991	0,9992	0,9992	0,9992	0,9992	0,9993	0,9993	0,0033
3,2	0,9993	0,9993	0,9994	0,9994	0,9994	0,9994	0,9994	0,9995	0,9995	0,9995	0,0024
3,3	0,9995	0,9995	0,9995	0,9996	0,9996	0,9996	0,9996	0,9996	0,9996	0,9997	0,0017
3,4	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9998	0,0012
3,5	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,0009
3,6	0,9998	0,9998	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,9999	0,0006

Formula sheet Probability Theory for BIT and TCS in module 4

Distribution	$\boldsymbol{E(X)}$	$\boldsymbol{var(X)}$
Geometric	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$n \cdot \frac{R}{N}$	$n \cdot \frac{R}{N} \cdot \frac{N-R}{N} \cdot \frac{N-n}{N-1}$
Poisson $P(X = x) = \frac{e^{-\mu}\mu^x}{x!}, x = 0, 1, 2, \ldots$	μ	μ
Uniform on (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang $f_X(x) = \frac{\lambda(\lambda x)^{n-1}e^{-\lambda x}}{(n-1)!}, x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$

$$var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} var(X_i) + \sum_{i} \sum_{\neq j} cov(X_i, X_j)$$