Probability theory, BIT & TCS, 1 July 2019.

1. (a) Not true. If X and Y are independent, then they are uncorrelated, but not v.v.

   (b) True. $X \sim B(250, 0.02)$

   $n > 25$ and $np = 5 < 10$ (large sample and small p)

   so X may be approximated with Poisson distr.

2. (a) $X | N=7 \sim B(7, 0.3)$

   $E(X | N=7) = 7 \cdot 0.3 = 2.1$

   (b) $P(X=2 \text{ and } N=7) = P(X=2 | N=7) \cdot P(N=7)$

   $= \binom{7}{2} (0.3)^2 (0.7)^5 \cdot \frac{6^7 e^{-6}}{7!}$

   $\approx 0.044 = 4.4\ \%$

   (c) $P(X=2) = \sum_{n=2}^{\infty} P(X=2 \text{ and } N=n)$

   $= \sum_{n=2}^{\infty} P(X=2 | N=n) \cdot P(N=n)$

   $= \sum_{n=2}^{\infty} \binom{n}{2} (0.3)^2 (0.7)^{n-2} \cdot \frac{6^n e^{-6}}{n!}$.

3. T: student passes the test

   H: student studies hard.

   Given: $P(T | H) = 0.80$, $P(T | \bar{H}) = 0.10$, $P(H) = 0.60$.

   $$P(H | T) = \frac{P(T|H) \cdot P(H)}{P(T|H) \cdot P(H) + P(T|\bar{H}) \cdot P(\bar{H})}$$ (Bayes' rule)

   $$= \frac{0.8 \cdot 0.6}{0.8 \cdot 0.6 + 0.1 \cdot 0.4} = \frac{0.48}{0.52} \approx 92.3\ \%$$

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4. (a) $\begin{cases} X: \text{number of participating members} \\ X \sim B(n, 0.8) \end{cases}$ (ok without motivation)

$\llcorner$ success prob. per person

$\llcorner$ total number of members.

(b) Here: $n = 6$. $6 - X \sim B(6, 0.2)$

$P(X \geq 5) = P(6 - X \leq 1) = 0.655$ (from table)

(c) By CLT X (with $n = 1000$) is approximately normal distributed with parameters $\begin{cases} \mu = 1000 \cdot 0.8 = 800 \\ \sigma^2 = 1000 \cdot 0.8 \cdot 0.2 = 160 \end{cases}$

So, $X \approx N(800, 160)$.

$$P(X \geq 780) \overset{\text{c.c.}}{=} P(X \geq 779.5)$$

$$\overset{\text{CLT}}{\approx} P\left(Z \geq \frac{779.5 - 800}{\sqrt{160}}\right) = P(Z \geq -1.62)$$

$$= \Phi(1.62) = 0.9474$$ (from table)

5. (a) $1 = \int_1^4 \frac{c}{t^3} dt = \left[-\frac{1}{2} c t^{-2}\right]_1^4 = -\frac{1}{2} c \left(\frac{1}{16} - 1\right) = c \cdot \frac{15}{32}$

$\Rightarrow c = 32/15.$

(b) $E(T) = \int_1^4 t \cdot \frac{32}{15} t^{-3} dt = \int_1^4 \frac{32}{15} t^{-2} dt = \left[-\frac{32}{15} t^{-1}\right]_1^4$

$= -\frac{32}{15}\left(\frac{1}{4} - 1\right) = \frac{8}{5} = 1.6$

$E(T^2) = \int_1^4 t^2 \cdot \frac{32}{15} t^{-3} dt = \int_1^4 \frac{32}{15} t^{-1} dt = \left[\frac{32}{15} \ln t\right]_1^4$

$= \frac{32}{15} \ln 4 \approx 2.957$

$\text{var}(T) = E(T^2) - (E(T))^2 = \frac{32}{15} \ln 4 - \left(\frac{8}{5}\right)^2 \approx 0.397.$

(c) $P(T \leq 3) = \int_1^3 \frac{32}{15} t^{-3} dt = \left[-\frac{16}{15} t^{-2}\right]_1^3 = -\frac{16}{15}\left(\frac{1}{9} - 1\right) = \frac{128}{135} \approx 0.948$

$= 94.8\,\%$

(d) $F_Y(y) = P(Y \leq y) = P(\sqrt{T} \leq y)$

$= P(T \leq y^2) = F_T(y^2)$

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_T(y^2) = 2y \cdot f_T(y^2)$

$$= \begin{cases} 2y \cdot \frac{32}{15} (y^2)^{-3} = \frac{64}{15} y^{-5}, & 1 \leq y^2 \leq 4 \Leftrightarrow 1 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

6. (a) X: amount of cereal in a box

$X \sim N(460, 6^2)$.

$P(X \leq 450) = P(Z \leq \frac{450-460}{6}) = \Phi(-1.67)$

$= 1 - \Phi(1.67) = 1 - 0.9525 = 0.0475$.

(b) $0.03 = P(X \leq 450) = P(Z \leq \frac{450-\mu}{6})$.

Solve for μ.

$P(Z \leq -\frac{450-\mu}{6}) = 0.97 \Leftrightarrow -\frac{450-\mu}{6} \approx 1.88$

$\Rightarrow \mu \approx 450 + 6 \cdot 1.88 = 461.28$ g

$\uparrow$ mean setting to be used.

(c) Assumptions: the weights of the boxes are independent and identically distributed.

X_{10}: total weight of 10 boxes. $X_{10} \sim N(10 \cdot 460, 10 \cdot 36)$

$= N(4600, 360)$ by independence

$P(4590 \leq X_{10} \leq 4610) = P(\frac{4590-4600}{\sqrt{360}} \leq Z \leq \frac{4610-4600}{\sqrt{360}})$

$= \Phi(0.53) - \Phi(-0.53) = 2\Phi(0.53) - 1$

$= 2 \cdot 0.7019 - 1 = 0.4038 \approx 40.4\%$