Kenmerk: EWI2019/TW/DMMP/MU/Mod7/Exam1

Exam 1, Module 7, Codes 201400483 & 201800141

Discrete Structures & Efficient Algorithms

Monday, March 18, 2019, 08:45 - 11:45

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4, both sides) per topic (ADS, DM).

This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	ca. 1h	(30 points)
Discrete Mathematics (DM)	ca. 2h	(60 points)

The total is 30+60=90 points. Your exam grade is the maximum of 1 and the total number of points divided by 9, rounded to one digit.

Important: It is necessary to use a new sheet of paper for each part (ADS and DM)!

Algorithms & Data Structures

1. (10 points)

   (a) Consider the following algorithm:

   ```
   def func(n):
       res=0

       while n>0:
           m=n
           while m>0:
               res=res+m
               m=m-1
           n=n-1

       return res
   ```

   Give the asymptotic time complexity of this algorithm, expressed in the number of arithmetical operations.

   (b) Suppose the number of steps of an algorithm, denoted by $T(n)$ for an input of size n, is given by the recurrence relation

   $$T(n) = 8T(n/2) + n^3 + 4n + 1/n$$

   What is the asymptotical complexity of this algorithm?

2. (5 points) Suppose in a heap E you change the key of an arbitrary element (suppose this element has index i). Give an algorithm *ExtHeapify(E,i)* that restores the heap property, if necessary.

3. (5 points) Given a binary tree with positive integer keys. Give an algorithm that determines whether the tree is sorted in order.

4. (10 points) A doctor has to see n patients. For each patient i the estimated time for seeing the patient is t_i minutes (an integer). The doctor should work at least T minutes, but wants to work as little time as possible. The overwork is the time the doctor works more than T minutes, so the objective is to minimize the overwork.

   We define a function $O(i, t)$ indicating the amount of overwork for an optimal choice from the patients $i, \ldots, n$ if the doctor should still work at least t minutes. Clearly $O(i, 0) = 0$ for $1 \leq i \leq n + 1$ (because then there is 0 minutes overwork, so optimal) and $O(n + 1, t) = \infty$ if $t > 0$ (because then there are no more patients for the t minutes the doctor should still work).

   (a) Motivate which of the following recurrence relations holds:
      i. $O(i, t) = min(t_i - t, O(i + 1, t))$
      ii. $O(i, t) = min(O(i + 1, t - t_i), O(i + 1, t))$
      iii. $O(i, t) = min(t_i - t, O(i + 1, t))$ if $t_i \geq t$, and
      $O(i, t) = min(O(i + 1, t - t_i), O(i + 1, t))$ otherwise

   (b) Give an algorithm to determine the minimum overwork time of the doctor. The complexity may be no worse that quadratic in n.

Discrete Mathematics

5. (10 points)

   (a) For given and fixed integer numbers $a, b \in \mathbb{Z}$, assume that we know that there exist $s, t, x, y \in \mathbb{Z}$ so that $as + bt = 250$ and $ax + by = 147$. Use the Euclidean algorithm to compute gcd$(250, 147)$ and use the result to show that a and b are relatively prime.

   (b) Let $a, b \in \mathbb{Z}$ with $a \geq b$. Prove that gcd$(a, b) =$ gcd$(a - b, b)$.

6. (10 points)

   (a) Denote by a_n the number of strings in $\{g, a, c, d\}^*$ of length n that do not contain an even number of a's. Compute a_1 and a_2, and set up a recurrence relation for a_n, for all $n \geq 3$. (You do not need to solve this recurrence relation.)

   (b) Compute the solution to the following recurrence relation.

   $$a_{n+2} = 10a_{n+1} - 25a_n + 2 \cdot 5^{n+1}, \text{ with } a_0 = 1 \text{ and } a_1 = 11\,.$$

7. (10 points)

   (a) Suppose we want to donate in total 160 Euros to 3 student associations R_1, R_2 and R_3, such that each student association gets at least 40 Euros, but at most 60 Euros. How many possibilities are there to do that? Use a generating function to compute your answer.

   (b) Would the answer be smaller, the same, or larger when asking for the number of possibilities to divide 160 Euros into three parts, with the same restrictions on minimal and maximal amounts? Justify your answer briefly (in once sentence, please).

8. (10 points) Let $G = (V, E)$ be a simple, undirected graph, and $\bar{G}$ be the complement graph of G (that has the same set of nodes as G and contains exactly all edges that are not in G). Show that, if both G and $\bar{G}$ are planar, then it is not possible that $|V| > 10$.

9. (14 points) Give a short proof or give a counterexample for each of the following claims.

   (a) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w(e) \geq 0$, $e \in E$. **Claim:** For any $s, t \in V$, there exists a minimum spanning tree that contains all edges of a shortest (s, t)-path.

   (b) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w(e) \geq 0$,$e \in E$. **Claim:** For any two minimum spanning trees T_1 and T_2, we have $\max\{w(e) \mid e \in T_1\} = \max\{w(e) \mid e \in T_2\}$.

   (c) Consider an undirected, simple graph $G = (V, E)$ with $s \in V$ fixed, and with edge weights $w(e) \geq 0$ such that $w_e \neq w_{e'}$ for all $e, e' \in E$ with $e \neq e'$. **Claim:** For all $v \in V$ there is a unique shortest (s, v)-path.

   (d) Consider a simple, capacitated network $G = (V, E, c)$, where E is the set of directed edges, and $c(e) \geq 0$, $e \in E$, are the edge capacities. **Claim:** Increasing the capacity of all edges of a minimum cut, will also increase the value of a maximum flow in the network.

10. (6 points) Consider a simple, capacitated network $G = (V, E, c)$, where V is the set of nodes, $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$ for $e \in E$ are the edge capacities. Suppose you are given a maximum (s, t)-flow f for G. With f given, suggest how to compute a minimum (s, t)-cut (S, T) for G.