Test Probability Theory, 26 July 2018, 13:45-15:45h (201300180)
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This test has 6 exercises, a formula sheet and tables of the binomial and normal distribution. You should motivate all answers. A regular scientific calculator is allowed, a programmable calculator ('GR') is not allowed.

1. A firm is bidding for a large construction project of a city. The management of the firm estimates the chance of winning the bid to be 60%. When all bids are received, the city's appraisal body, which allocates the project, can ask for extra information. From the past it is known that in 75% of the winning bids extra information was asked and that in 40% of the non-winning bids extra information was asked.

   During the bid procedure the firm is asked for extra information. Use this information to calculate the probability that the firm wins the bid. First, define a number of relevant events and write down the given probabilities in terms of (conditional) probabilities of those events.

2. Person C claims to be clairvoyant. To test this, 10 non-transparant boxes are shown to him. Each box contains a bottle that is randomly filled with either oil or water, with equal probabilities for both options. Without opening the boxes, person C has to say for each of the 10 boxes if the bottle in the box contains oil or water.

   Let X be the number of correct answers. Assume that person C is not clairvoyant at all and thus randomly selects an answer for each box.

   (a) Calculate $P(X \geq 8)$.
   (b) Calculate $E(X)$.
   (c) Calculate $var(X)$.

3. The joint probability distribution $P(X = x$ and $Y = y)$ of the random variables $X$and $Y$is given in the table below.

y / x	0	1	4
-1	0.04	0.10	0.15
0	0.16	0.05	0.10
1	0.20	0.10	0.10

   (a) Determine the marginal probability distributions of X and Y.
   (b) Calculate the covariance of X and Y, $cov(X, Y)$.
   (c) Calculate $P(X^2 + Y = 1)$.
   (d) Calculate $E(Y|X = 1)$.

4. Suppose the continuous random variable X is uniformly distribution on the interval $[a, b]$ and has mean 1 and variance 4/3.

   (a) Show that $a = -1$ and $b = 3$.

   (b) Determine the 10th percentile of X.

   (c) Determine the density function of $Y = X^3$.

5. During a particular period a university's information technology office received 20 service orders for problems with laptops, of which 8 were Macs and 12 were HPs. A sample of 6 of these service orders is to be selected for inclusion in a customer satisfaction survey. Suppose that the six are selected in a completely random fashion, so that any particular subset of size six has the same chance of being selected as does any other subset. Let X be the number of selected service orders for HP laptops.

   (a) Explain why X follows a hypergeometric distribution.

   (b) Calculate $P(X = 2)$, the probability that exactly two of the selected service orders were for HP laptops.

   Next, consider another (longer) period of time in which the office received 4800 service orders for problems with laptops, of which 1920 were Macs and 2880 were HPs. Let Y be the number of selected service orders for HP laptops in a sample of size 30 in this period.

   (c) Explain why we can approximate the distribution of Y with the binomial distribution.

   (d) Calculate or approximate $P(Y \leq 12)$, the probability that at most 12 of the 30 selected service orders were for HP laptops.

6. Two types of customers arrive at a ticket window for service. The corresponding service times X and Y can be modelled as independent and exponentially distributed random variables with parameters $\lambda = 1$ and $\lambda = 2$ respectively.

   (a) Determine $E(X + Y)$ and $var(X + Y)$.

   (b) Calculate $\rho(X, X + Y)$.

   (c) On a certain day 100 customers of each type arrive at the ticket window. Assume that 100 service times are a random sample X_1, X_2, ..., X_{100} of X (so, these are 100 independent and $Exp(1)$-distributed service times) and the other 100 service times are a random sample Y_1, Y_2, ..., Y_{100} of Y. Which probability distribution, including parameter values, does the total service time of the 200 customers have, approximately? Explicitly mention the properties that you use.

Points:

1	2			3				4			5				6			total
	a	b	c	a	b	c	d	a	b	c	a	b	c	d	a	b	c	
4	2	1	1	2	3	2	2	1	2	3	2	2	1	4	2	2	3	39

Grade: $\dfrac{\text{number of points}}{39} \times 9 + 1$ (rounded to one decimal)

Formula sheet Probability Theory for BIT and TCS in module 4

Distribution	$\boldsymbol{E(X)}$	$\boldsymbol{var(X)}$
Geometric	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$n \cdot \frac{R}{N}$	$n \cdot \frac{R}{N} \cdot \frac{N-R}{N} \cdot \frac{N-n}{N-1}$
Poisson $\quad P(X=x) = \frac{e^{-\mu}\mu^x}{x!},\ x = 0, 1, 2, \ldots$	μ	μ
Uniform on $[a, b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang $\quad f_X(x) = \frac{\lambda(\lambda x)^{n-1}e^{-\lambda x}}{(n-1)!},\ x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$

$$var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} var(X_i) + \sum\sum_{i \neq j} cov(X_i, X_j)$$