Linear Algebra Date : April 01, 2022
Time : 13.45 – 15.45 hrs

1. a) Determine A^0, A^2, and A^T
   $A^0 = I$
   $A^2 = \begin{pmatrix} 4 & 8 \\ 0 & 4 \end{pmatrix}$
   $A^T = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$

   b) Find all the eigenvalues of $A^{-1} + A$

   $$\lambda = 5/2$$

2. $(5v_3 = v_2 + 2v_1)$
   The dimension of $NullA$ is 2

3. a) A has an eigenvalue 1 for values of α : $\frac{-3}{2}, \frac{3}{2}$

   b) If $\alpha = 0$, then the eigenspace of A corresponding to value 4 is:

   $$E_4 = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

4. Match the transformations with either of the matrices from A to G:

Transformations →	T_1	T_2	T_3	T_4	T_5
	C	F	D	B	G

5. Bring the system to the echelon form:

$$\begin{pmatrix} \alpha & \alpha^2 & 2 & \alpha^2 \\ 1 & \alpha-1 & \alpha & 0 \\ 1 & -1 & 2\alpha & \alpha \end{pmatrix} \sim \ldots \sim \begin{pmatrix} 1 & \alpha-1 & \alpha & 0 \\ 0 & \alpha & -\alpha & -\alpha \\ 0 & 0 & (\alpha-2)(\alpha+1) & -\alpha(\alpha+1) \end{pmatrix}$$

The intersection in a line corresponds to one free variable,

(i) x_2 is a free variable only if

- $\alpha = 0, (\alpha-2)(\alpha+1) \neq 0$
- $\alpha = 0, \alpha \neq 2, \alpha \neq -1$

(ii) x_3 is a free variable only if

- $\alpha \neq 0, (\alpha-2)(\alpha+1) = 0, -\alpha(\alpha+1) = 0$
- $\alpha \neq 0, \alpha \neq 2, \alpha = -1$ (note that $\alpha = 2$ gives an inconsistent system).

6. a) No, S_a is not a subspace of V. Take, for instance, the zero vector $(0, 0)$.
   Since $2(0) - 3(0) \neq 6$
   $\mathbf{0} \notin S_a$, hence S_a is not a subspace of V.

b) Yes, S_b is a subspace of V.
To prove this,

i) take $\mathbf{0}$, because A.$\mathbf{0}$=3.$\mathbf{0}$, we have $\mathbf{0} \in S_b$.
It is also sufficient to show that S_b is a non empty set.

ii) Suppose $\mathbf{x}_1 \in \mathbf{S}_b$ and $\mathbf{x}_2 \in \mathbf{S}_b$.
This means that

$$A\mathbf{x}_1 = 3\mathbf{x}_1, A\mathbf{x}_2 = 3\mathbf{x}_2$$

Hence

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A(\mathbf{x}_1) + A(\mathbf{x}_2) = 3\mathbf{x}_1 + 3\mathbf{x}_2 = 3(\mathbf{x}_1 + \mathbf{x}_2),$$

meaning that $\mathbf{x}_1 + \mathbf{x}_2 \in \; S_b$ as well.

iii) Moreover, for any $c \in \mathbb{R}, A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(3\mathbf{x}_1) = 3(c\mathbf{x}_1)$ meaning that $c\mathbf{x}_1 \in S_b$.

We have shown above that S_b is non-empty set which is closed under addition and closed under scalar multiplication. Hence S_b is a subset of $V = \mathbb{R}^n$.

7. We know that **u** and **v** form a basis for $\mathcal{U}$
and reducing the augmented matrix: (**u v** | **w**)
$\mathbf{w} = 7\mathbf{u} - 4\mathbf{v}$, that is **w** is a linear combination of **u** and **v**.
Hence **w** is in $\mathcal{U}$
Therefore $[w]_B = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$

8. We have P=$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and D=$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
The matrix A is related to $P^{-1}AP$ as follows: $A = PDP^{-1}$.
Obtain P^{-1}=$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

$A = PDP^{-1}$=$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$

Alternatively:

A$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$=(-1)$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$=$\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

A$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$= A$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$-A$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$=$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$-$\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$=$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

A$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$= A$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$-A$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$=$\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$-$\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$=$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

Hence we obtain A = $\begin{pmatrix} -1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$

9. a) We can show that

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{-1}{2}(T\begin{pmatrix}0\\-3\\0\end{pmatrix} + T\begin{pmatrix}-2\\3\\0\end{pmatrix}) = \frac{-1}{2}(\begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}2\\0\end{pmatrix}) = \begin{pmatrix}-3\\-1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{-1}{3}T\begin{pmatrix}0\\-3\\0\end{pmatrix} = \frac{-1}{3}\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-4/3\\-2/3\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{5}{6}T\begin{pmatrix}0\\-3\\0\end{pmatrix} + \frac{1}{6}T\begin{pmatrix}-2\\3\\0\end{pmatrix} + \frac{1}{3}T\begin{pmatrix}1\\6\\3\end{pmatrix} = \frac{5}{6}\begin{pmatrix}4\\2\end{pmatrix} + \frac{1}{6}\begin{pmatrix}2\\0\end{pmatrix} + \frac{1}{3}T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}11/3\\2\end{pmatrix}$$

Hence we have the representation matrix:
$A = T(\boldsymbol{e_1 e_2 e_3}) = \begin{pmatrix}-3 & -4/3 & 11/3\\-1 & -2/3 & 2\end{pmatrix}$

b) The reduced echelon form of A is $\begin{pmatrix}-3 & -4/3 & 11/3\\-1 & -2/3 & 2\end{pmatrix} \sim \dots \sim \begin{pmatrix}-1 & 0 & 1/3\\0 & 1 & -7/2\end{pmatrix}$
We have Null $A \neq \{\mathbf{0}\}$ hence T is NOT one-to-one.
Col(A)=

$$imT = \text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

OR

$$imT = \text{Span}\left\{\begin{pmatrix}-3\\-1\end{pmatrix}, \begin{pmatrix}-4/3\\-2/3\end{pmatrix}\right\}$$

$imT = \mathbb{R}^2$ hence T is onto.