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Department of Electrical Engineering, Mathematics and Computer Science

Solution exam Linear Algebra on Tuesday July 21, 2020, 18.15 – 20.15 hours.

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2. Consider the following system of equations:

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 2 \\ \beta x_2 + x_3 + x_4 = 0 \\ \alpha x_3 = 0 \end{cases}$$

We known that the solution set of the system is given by:

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Determine all possible α and β for which the above is correct.

First we check whether the given solution satisfies our linear system of equations. First we check whether

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

satisfies the given system of equations. It is easily seen that this yields that $\beta = 1$. Next we check whether

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

satisfies the homogeneous version of our system of equations:

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 0 \\ \beta x_2 + x_3 + x_4 = 0 \\ \alpha x_3 = 0 \end{cases}$$

and it easily see that this is the case for $\beta = 1$. We still have not found any restriction on α. However, we have not checked whether there are additional solutions. The augmented matrix of the system:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & \beta & 1 & 1 & 0 \\ 0 & 0 & \alpha & 0 & 0 \end{pmatrix}$$

is already in the echelon form (given $\beta = 1$). However if $\alpha = 0$ then we have only two pivots and two free variables and hence we will find additional solutions. Therefore, we will find our solution set (with one free variable) only if $\alpha \neq 0$.

3.

For which values of α is the matrix:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \alpha \\ 0 & \alpha & 1 \end{pmatrix}$$

diagonalizable.

Let us compute the eigenvalues of this matrix. The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 1 & -\alpha \\ 0 & -\alpha & \lambda - 1 \end{pmatrix} = (\lambda - 1)\left[(\lambda - 1)^2 - \alpha^2\right]$$

The eigenvalues are the zeros of this polynomial and hence equal to 1, $1 - \alpha$ and $1 + \alpha$. Clearly the eigenvalues are all distinct for $\alpha \neq 0$. If all eigenvalues are distinct then it is known that the matrix is diagonalizable.

For $\alpha = 0$ we find:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we find that we have a triple zero of the characteristic polynomial in 0. Therefore, the matrix is diagonalizable if we can find three independent eigenvectors. However, if we solve

$$(I - A)\mathbf{x} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

then we find:

$$\mathbf{x} \in \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

and hence we can only find two independent eigenvectors. Therefore for $\alpha = 0$ the matrix is not diagonalizable.

4.

Given are matrices A and B and C

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

Find all matrices X such that $AX - XB = C$.

Let

$$X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

Working out $AX - XB = C$ we then find:

$$\begin{aligned} -x_1 + x_2 - x_3 &= -1 \\ x_4 &= 1 \end{aligned}$$

We find the following solution set:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \right\}$$

5.

Find all possible α for which the volume of the parallelepiped with vertices $(0,0,0)$, $(0,1,0)$, $(\alpha,0,\alpha)$ and $(2,0,\alpha)$ is equal to 3.

The volume of the parallelepiped is given by:

$$\left| \det \begin{pmatrix} 0 & \alpha & 2 \\ 1 & 0 & 0 \\ 0 & \alpha & \alpha \end{pmatrix} \right| = |-\alpha^2 + 2\alpha|$$

We need either:

$$-\alpha^2 + 2\alpha = 3$$

or

$$-\alpha^2 + 2\alpha = -3$$

The first one does not yield any solutions. The second one yields $\alpha = -3$ or $\alpha = 1$.

6.

Consider an invertible matrix $A \in \mathbb{R}^{n \times n}$. a) If λ is an eigenvalue of A show that λ^{-1} is an eigenvalue of A^{-1}.

If λ is an eigenvalue of A then clearly $\lambda \neq 0$ (matrix is invertible). Moreover there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} = \lambda \mathbf{x}$$

multiplying the equation with $\lambda^{-1}A^{-1}$ on the left we obtain:

$$\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$$

but this means that λ^{-1} is an eigenvalue of A^{-1}.

b) Given is that the matrix $B = A^3 - 2A^2$ is invertible. Show that A does **not** have eigenvalue 2

Assume 2 is an eigenvalue of A. Then there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} = 2\mathbf{x}$$

but then:

$$B\mathbf{x} = \left(A^3 - 2A^2\right)\mathbf{x} = (8-8)\mathbf{x} = \mathbf{0}$$

But since $\mathbf{x} \neq \mathbf{0}$ this yields a contradiction with B invertible. Hence A does **not** have eigenvalue 2

7.

> We have the following matrix:
>
> $$A = \begin{pmatrix} \alpha & \alpha+\beta-1 & -1 \\ 2-\alpha & 1-\beta-\alpha & -1 \\ \alpha & \alpha-2 & -1 \end{pmatrix}$$
>
> We know that a basis for $\operatorname{Null} A$ is given by:
>
> $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$
>
> while a basis for $\operatorname{Col} A$ is given by:
>
> $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$
>
> Determine α and β.

Given our basis for $\operatorname{Null} A$, we know that we must have:

$$A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}$$

This yields $\alpha = 1$ and hence:

$$A = \begin{pmatrix} 1 & \beta & -1 \\ 1 & -\beta & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

It is easily verified that the first and third column of A are in the given $\operatorname{Col} A$. However, the second column of A is only in the given $\operatorname{Col} A$ if $\beta = -1$. We find:

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

It is easily verified that this matrix has the required $\operatorname{Col} A$ and $\operatorname{Null} A$.

8.

> $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which first mirrors each point $(x_1, x_2) \in \mathbb{R}^2$ in the line $y = x$ and next rotates around the origin over α radians (counterclockwise). The representation matrix of T is given by:
>
> $$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
>
> Determine $\alpha \in [0, 2\pi)$.

Let's first consider $(1,0)$. After the mirroring this is in $(0,1)$. The vector $(0,1)$ should then be rotated α radians counterclockwise and (according to the representation matrix) end up in $(1,0)$. It is easily seen that this is a rotation over $3\pi/2$ radians.

Let's also consider $(0,1)$. After the mirroring this is in $(1,0)$. The vector $(1,0)$ should then be rotated α radians counterclockwise and (according to the representation matrix) end up in $(0,-1)$. It is easily seen that this is also a rotation over $3\pi/2$ radians.

Therefore $\alpha = 3\pi/2$.

9.

Is the set $S = \{(x,y,z) \in \mathbb{R}^3 \mid (x-z)(y-z) = 0\}$ a subspace of $\mathbb{R}^3$?

For a subspace we have two requirements: If $\mathbf{x}_1$ and $\mathbf{x}_2$ are in the set S then we should have:

$$\lambda \mathbf{x}_1 \in S$$

for all λ and

$$\mathbf{x}_1 + \mathbf{x}_2 \in S.$$

The first property is satisfied in this case. However, the second property is not satisfied. For instance:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are both in S but their sum:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

is clearly not is S. Therefore it is not a subspace.